

RATIONAL ORBITS OF THE SPACE OF PAIRS OF EXCEPTIONAL JORDAN ALGEBRAS

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ABSTRACT. Let k be a field of characteristic not equal to 2, 3, \mathcal{O} an octonion over k and \mathcal{J} the exceptional Jordan algebra defined by \mathcal{O} . We consider the prehomogeneous vector space (G, V) where $G = GE_6 \times GL(2)$ and $V = \mathcal{J} \oplus \mathcal{J}$. We prove that generic rational orbits of this prehomogeneous vector space are in bijective correspondence with k -isomorphism classes of pairs $(\mathcal{M}, \mathfrak{n})$ where \mathcal{M} 's are isotopes of \mathcal{J} and \mathfrak{n} 's are cubic étale subalgebras of \mathcal{M} . Also we prove that if \mathcal{O} is split, then generic rational orbits are in bijective correspondence with isomorphism classes of separable extensions of k of degrees up to 3.

1. INTRODUCTION

Part of the results in this paper is taken from the first author's master thesis.

Let k be a field of characteristic not equal to 2, 3, $k^\times = k \setminus \{0\}$, k^{sep} the separable closure of k and \bar{k} the algebraic closure of k . We use the notation $\text{ch}(k)$ for the characteristic of k . If X is a finite set, then let $|X|$ denote its cardinality. We denote the algebra of $n \times n$ matrices by $M(n)$ and the group of $n \times n$ invertible matrices by $GL(n)$. Let $SL(n) = \{g \in GL(n) \mid \det(g) = 1\}$. If V is a finite dimensional vector space over k , then $GL(V)$ is the group of invertible k -linear maps from V to V itself. For $g \in GL(V)$, $\det(g)$ is well-defined. Let $SL(V) = \{g \in GL(V) \mid \det(g) = 1\}$. We denote the Lie algebras of $GL(n)$, $GL(V)$ by $\mathfrak{gl}(n)$, $\mathfrak{gl}(V)$ respectively.

Let $\tilde{\mathcal{O}}$ be the split octonion over k . It is the normed algebra over k obtained by the Cayley–Dickson process (see [4, pp.101–110]). If $A = M(2)$ and the norm is the determinant, then $\tilde{\mathcal{O}}$ is $A(+)$ with the notation of [4]. An *octonion* is, by definition, a normed algebra which is a k -form of $\tilde{\mathcal{O}}$. Let \mathcal{O} be an octonion. We use the notation $\|x\|$ for the norm of $x \in \mathcal{O}$. If $a \in k$, $\|ax\| = a^2\|x\|$. For $x, y \in \mathcal{O}$, let

$$Q(x, y) = \frac{1}{2}(\|x + y\| - \|x\| - \|y\|).$$

This is a non-degenerate symmetric bilinear form such that $Q(x, x) = \|x\|$. Note that $2Q(x, y)$ is denoted by $\langle x, y \rangle$ in [12, p.1]. Let $W \subset \mathcal{O}$ be the orthogonal complement of $k \cdot 1$ with respect to Q . If $x = x_1 + x_2$ where $x_1 \in k \cdot 1$, $x_2 \in W$, then we define

$$(1.1) \quad \iota(x) = \bar{x} = x_1 - x_2$$

2000 *Mathematics Subject Classification.* 11S90, 11R45.

Key words and phrases. prehomogeneous vector spaces, Jordan algebra, cubic fields.

The first author was partially supported by Grant-in-Aid (B) (24340001)

and call it the *conjugate* of x . Note that $\|x\| = x\bar{x}$. The map $\mathcal{O} \ni x \mapsto \iota(x) \in \mathcal{O}$ is an element of $\mathrm{GL}(\mathcal{O})$ where \mathcal{O} is regarded as a k -vector space. For $x \in \mathcal{O}$, we define the trace $\mathrm{tr}(x)$ by $\mathrm{tr}(x) = x + \bar{x}$. Note that

$$\mathrm{tr}(x\bar{y}) = \mathrm{tr}(\bar{y}x) = 2Q(x, y), \quad \mathrm{tr}(xy) = \mathrm{tr}(yx).$$

Also the following properties:

$$(1.2) \quad \mathrm{tr}((xy)z) = \mathrm{tr}(x(yz))$$

are satisfied for all $x, y, z \in \mathcal{O}$ (see [12, p.8, Lemma 1.3.2]). Therefore, we may write $\mathrm{tr}(xyz)$ instead of $\mathrm{tr}((xy)z)$ or $\mathrm{tr}(x(yz))$.

Let \mathcal{J} be the exceptional Jordan algebra over k . Any element $X \in \mathcal{J}$ is of the form:

$$(1.3) \quad X = \begin{pmatrix} s_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & s_2 & x_1 \\ x_2 & \bar{x}_1 & s_3 \end{pmatrix}, \quad s_i \in k, \quad x_i \in \mathcal{O} \quad (i = 1, 2, 3).$$

We sometimes denote this element by $h(s_1, s_2, s_3, x_1, x_2, x_3)$. For elements of \mathcal{J} , the notion of the determinant is well-defined and is given by

$$(1.4) \quad \det(X) = s_1 s_2 s_3 + \mathrm{tr}(x_1 x_2 x_3) - s_1 \|x_1\| - s_2 \|x_2\| - s_3 \|x_3\|.$$

The multiplication of \mathcal{J} is defined as follows:

$$X \circ Y = \frac{1}{2}(XY + YX),$$

where the multiplication used on the right-hand side is the multiplication of matrices.

Let \mathfrak{S}_n be the symmetric group of $\{1, \dots, n\}$. We define the multiplication of $\sigma, \tau \in \mathfrak{S}_n$ by

$$(1.5) \quad (\sigma\tau)(i) = \tau(\sigma(i))$$

for $i \in \{1, \dots, n\}$. Let $(i\ j)$ denote the transposition of $i \neq j \in \{1, \dots, n\}$.

The algebraic groups $\mathrm{SO}(Q)$, E_6 and GE_6 are given by

$$\mathrm{SO}(Q) = \{\alpha \in \mathrm{SL}(\mathcal{O}) \mid \forall x, y \in \mathcal{O}, Q(\alpha(x), \alpha(y)) = Q(x, y)\},$$

$$E_6 = \{L \in \mathrm{GL}(\mathcal{J}) \mid \forall X \in \mathcal{J}, \det(LX) = \det(X)\},$$

$$GE_6 = \{L \in \mathrm{GL}(\mathcal{J}) \mid \forall X \in \mathcal{J}, \det(LX) = c(L) \det(X) \text{ for some } c(L) \in \mathrm{GL}(1)\}$$

respectively. Then $c : GE_6 \rightarrow \mathrm{GL}(1)$ is a character and there exists an exact sequence

$$(1.6) \quad 0 \rightarrow E_6 \hookrightarrow GE_6 \xrightarrow{c} \mathrm{GL}(1) \rightarrow 0.$$

It is known that E_6 is a smooth connected quasi-simple simply-connected algebraic group of type E_6 (see [12, p.181, Theorem 7.3.2]). The terminology “quasi-simple” means that its inner automorphism group is simple (see [11, p.136]). Since the dimension of E_6 as a variety and the dimension of the Lie algebra of E_6 coincide (see the proof of [12, p.181, Theorem 7.3.2]), the smoothness of the group follows.

Let $H_1 = E_6$, $G_1 = GE_6$ and $H = H_1 \times \mathrm{GL}(2)$ respectively. Let

$$(1.7) \quad G = G_1 \times \mathrm{GL}(2), \quad V = \mathcal{J} \otimes \mathrm{Aff}^2$$

where Aff^2 is the 2-dimensional affine space regarded as a vector space. Then V is a representation of G .

We define a character c' of G by the composition of homomorphisms

$$(1.8) \quad G = G_1 \times \text{GL}(2) \xrightarrow{\text{pr}_1} G_1 \xrightarrow{c} \text{GL}(1)$$

where “ pr_1 ” is the natural projection. The representation \mathcal{J} of G_1 is irreducible (see the proof of [12, p.181, Theorem 7.3.2]) and so V is an irreducible representation of G . The pair (G, V) is what we call a prehomogeneous vector space.

We review the definition of prehomogeneous vector spaces. In the following definition, k is an arbitrary field and G, V are not necessarily the above G, V .

Definition 1.9. Let G be a connected reductive group, V a representation and χ a non-trivial primitive character of G , all defined over k . Then, (G, V, χ) is called a *prehomogeneous vector space* if it satisfies the following properties.

- (i) There exists a Zariski open orbit.
- (ii) There exists a non-constant polynomial $\Delta(x) \in k[V]$ such that $\Delta(gx) = \chi(g)^a \Delta(x)$ for a positive integer a .

The polynomial Δ is called a *relative invariant polynomial*.

If (G, V, χ) is an irreducible (as a representation) prehomogeneous vector space, then the choices of χ, Δ are essentially unique and we may write (G, V) instead of (G, V, χ) . We define $V_k^{\text{ss}} = \{x \in V_k \mid \Delta(x) \neq 0\}$ and call it the set of *semi-stable points*.

Now we assume that $\text{ch}(k) \neq 2, 3$ and that G, V are as in (1.7) again. We shall show the existence of an open orbit in Proposition 3.2(2).

We identify Aff^2 as the space of linear forms in two variables $v = (v_1, v_2)$ and regard elements of V as the set of $x = x_1 v_1 + x_2 v_2$ where $x_1, x_2 \in \mathcal{J}$. The action of $g = (g_1, g_2) \in G$ where $g_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on V is given by

$$(1.10) \quad g(x_1 v_1 + x_2 v_2) = g_1(x_1)(a v_1 + c v_2) + g_1(x_2)(b v_1 + d v_2).$$

For $x \in V$, let

$$(1.11) \quad F_x(v) = F_x(v_1, v_2) = \det(x_1 v_1 + x_2 v_2).$$

This is a homogeneous polynomial of degree 3 of $v = (v_1, v_2)$. We define an action of G on $\text{Sym}^3 \text{Aff}^2$ by

$$(1.12) \quad G \times \text{Sym}^3 \text{Aff}^2 \ni ((g_1, g_2), f(v)) \mapsto c(g_1) f(v g_2) \in \text{Sym}^3 \text{Aff}^2,$$

where we are regarding v as a row vector. Then the map $V \ni x \mapsto F_x \in \text{Sym}^3 \text{Aff}^2$ is G -equivariant.

For $f \in \text{Sym}^3 \text{Aff}^2$, we denote the discriminant of f by $\Delta(f)$. It is easy to see that $\Delta(F_x)$ is a homogeneous polynomial of degree 12 in $k[V]$, and that

$$\Delta(F_{gx}) = c(g_1)^4 (\det(g_2))^6 \Delta(F_x).$$

So (G, V) is an irreducible prehomogeneous vector space and $\Delta(F_x)$ is a relative invariant polynomial of degree 12 of x . We shall discuss the details in Section 3.

Let

$$(1.13) \quad w = w_1 v_1 + w_2 v_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} v_1 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} v_2 \in \mathcal{J} \otimes \text{Aff}^2.$$

Let G_w denote the stabilizer of w in G and $H_w = G_w \cap H$.

Suppose that (G, V) is an arbitrary irreducible prehomogeneous vector space over k . If there exists $w \in V_k^{\text{ss}}$ such that G_w is reductive (we are assuming that reductive groups are smooth over the ground field), then (G, V) is said to be *regular*. If (G, V) is irreducible and regular, it is known that $V_{k^{\text{sep}}}^{\text{ss}}$ is a single $G_{k^{\text{sep}}}$ -orbit (see [15]). Note that it is proved in [15] that $V_{k^{\text{sep}}} \setminus G_{k^{\text{sep}}} w$ is a hypersurface. However, since the representation is irreducible, the k -algebra generated by relative invariant polynomials has a single generator, say $\Delta(x)$. Then $V_{k^{\text{sep}}} \setminus G_{k^{\text{sep}}} w$ must coincide with $\{x \in V_{k^{\text{sep}}} \mid \Delta(x) = 0\}$. Therefore, $\{x \in V_{k^{\text{sep}}} \mid \Delta(x) \neq 0\} = G_{k^{\text{sep}}} w$ and so $V_{k^{\text{sep}}}^{\text{ss}}$ is a single $G_{k^{\text{sep}}}$ -orbit.

This implies that for any $x \in V_{k^{\text{sep}}}^{\text{ss}}$, there exists $g \in G_{k^{\text{sep}}}$ such that $x = gw$. We define a map c_x by

$$(1.14) \quad c_x : \text{Gal}(k^{\text{sep}}/k) \ni \sigma \mapsto g^{-1} g^\sigma \in G_{w k^{\text{sep}}}.$$

Then c_x is a 1-cocycle and defines an element of the first Galois cohomology set $H^1(k, G_w)$. For the definition of the Galois action and the Galois cohomology set, see Section 2. For arbitrary algebraic groups G, H defined over k , a homomorphism $f : G \rightarrow H$ defined over k and a map $h_f^1 : H^1(k, G) \rightarrow H^1(k, H)$ which is induced by f , we denote the inverse image of $1 \in H^1(k, H)$ under h_f^1 by $\text{Ker}(H^1(k, G) \rightarrow H^1(k, H))$. The following theorem is well-known (see [5, pp.268,269] for example).

Theorem 1.15. *If (G, V) is an irreducible prehomogeneous vector space, the orbit of $w \in V$ is open in V and G_w is reductive, then the map*

$$(1.16) \quad G_k \backslash V_k^{\text{ss}} \ni x \mapsto c_x \in \text{Ker} \left(H^1(k, G_w) \rightarrow H^1(k, G) \right)$$

is well-defined and bijective.

Note that it is assumed in [5] that $\text{ch}(k) = 0$. The proof of the above theorem works as long as $V_{k^{\text{sep}}}^{\text{ss}}$ is a single $G_{k^{\text{sep}}}$ -orbit. As we pointed out above, with the assumption of the theorem, $V_{k^{\text{sep}}}^{\text{ss}}$ is a single $G_{k^{\text{sep}}}$ -orbit even if the characteristic of k is positive. So the above theorem holds in all characteristics.

Now we go back to our situation and assume that $G = G_1 \times \text{GL}(2)$, $V = \mathcal{J} \otimes \text{Aff}^2$. When k is the field of complex numbers, Sato and Kimura have shown in [10, pp.138,139] that (G, V) is a prehomogeneous vector space having a relative invariant polynomial of degree 12 and for a generic point $w \in V$, the Lie algebra of G_w is isomorphic to the Lie algebra of $\text{SO}(Q)$. However, the structure of G_w has not been determined completely over an arbitrary ground field and the interpretation of $G_k \backslash V_k^{\text{ss}}$ is unknown.

In Section 3, we determine the structure of the stabilizer as an algebraic group over k and obtain the following proposition.

Proposition 1.17. *G_w is isomorphic to $\mathrm{GL}(1) \times (\mathrm{Spin}(Q) \rtimes \mathfrak{S}_3)$. In particular, G_w is a smooth reductive algebraic group.*

Let $\mathrm{JIC}(k)$ be the set of equivalence classes of pairs $(\mathcal{M}, \mathfrak{n})$ where \mathcal{M} 's are isotopes of \mathcal{J} and \mathfrak{n} 's are cubic étale subalgebras of \mathcal{M} (see Section 4 or [12, pp.154–158] for the definition of isotopes of \mathcal{J}). For the details of the equivalence relation, see Section 4. In Section 4, we associate a pair $(\mathcal{M}, \mathfrak{n}) \in \mathrm{JIC}(k)$ to each point in V_k^{ss} . Our main theorem concerns a correspondence between the set $G_k \backslash V_k^{\mathrm{ss}}$ of rational orbits and the set of equivalence classes of pairs $(\mathcal{M}, \mathfrak{n})$ as above. The following theorem is our main theorem.

Theorem 1.18. *The set $G_k \backslash V_k^{\mathrm{ss}}$ corresponds bijectively with the set $\mathrm{JIC}(k)$.*

In Section 5, to each element of V_k^{ss} , we associate an isotope of \mathcal{J} and its cubic étale subalgebra explicitly by constructing an equivariant map from $\mathcal{J} \otimes \mathrm{Aff}^2$ to \mathcal{J} .

If $\alpha \in \bar{k}$ and $F_x(\alpha, 1) = 0$, we call α a root of $F_x(v)$. In Section 6, we prove the following theorem.

Theorem 1.19. *If k is a finite field or \mathbb{O} is the split octonion, then there is a bijective correspondence between $G_k \backslash V_k^{\mathrm{ss}}$ and $H^1(k, \mathfrak{S}_3)$. Moreover, if $x \in V_k^{\mathrm{ss}}$, then the corresponding cohomology class in $H^1(k, \mathfrak{S}_3)$ is the element determined by the action of the Galois group on the set of roots of $F_x(v)$.*

If $n > 0$, then it is well-known that \mathfrak{S}_n is the automorphism group of the k -algebra k^n . Therefore, the set $H^1(k, \mathfrak{S}_n)$ can be identified with k -isomorphism classes of étale k -algebras of degree n . In the situation of Theorem 1.19, considering the case $n = 3$, rational orbits in V_k^{ss} are in bijective correspondence with separable extensions of k of degrees up to 3. Also if $x \in V_k^{\mathrm{ss}}$ corresponds to a separable cubic extension, then the identity component G_w° is isomorphic to a k -form of $\mathrm{Spin}(Q)$ which comes from the triality and so $G_k \backslash V_k^{\mathrm{ss}}$ parametrizes such objects.

The problem of finding arithmetic interpretations of rational orbits of prehomogeneous vector spaces is classical and goes back to the work of Gauss. Some cases which are in some sense similar to our case have been considered. For example, the prehomogeneous vector spaces considered in [14], [6], [13] have close relations with field extensions of the ground field of degrees up to 5. Among the cases considered in these papers, consider the following prehomogeneous vector spaces:

- (a) $G = \mathrm{GL}(3) \times \mathrm{GL}(2)$, $V = \mathrm{Sym}^2 \mathrm{Aff}^3 \otimes \mathrm{Aff}^2$ (see [14]),
- (b) $G = \mathrm{Res}_{k_1/k} \mathrm{GL}(3) \times \mathrm{GL}(2)$, $V = H_3(k_1) \otimes \mathrm{Aff}^2$ (see [6]),
- (c) $G = \mathrm{GL}_3(D) \times \mathrm{GL}(2)$, $V = H_3(D) \otimes \mathrm{Aff}^2$ (see [13])

where k_1 is a quadratic extension of k , D is a quaternion algebra over k and $H_3(k_1), H_3(D)$ are the spaces of 3×3 Hermitian matrices with entries in k_1, D respectively. These cases have the same set of weights with respect to their maximal split tori. If the ground field is \mathbb{R} , these cases and our case correspond to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ (the Hamiltonian), \mathbb{O} . So our case is the last of these series.

The problem of finding arithmetic interpretations of rational orbits of our case over rings is more difficult. We may consider such problems in the future.

In Section 2, we define and review basic notions such as the Galois cohomology, the Lie algebra of groups of type E_6 and the triality concerning $\text{Spin}(8)$. In Section 3, we determine the structure of the stabilizer G_w . In Section 4, we give an intrinsic arithmetic interpretation of the set of rational orbits $G_k \backslash V_k^{\text{ss}}$. In Section 5, we construct the isotopes corresponding to points in V_k^{ss} explicitly by using an equivariant map from V to \mathcal{J} . In Section 6, we prove Theorem 1.19.

2. PRELIMINARIES

In this section, we define basic notations and prove fundamental facts used in the subsequent sections.

We define an action of the Galois group on an algebraic group as follows. Suppose that K is a Galois extension of k . Then we define the multiplication in the Galois group $\text{Gal}(K/k)$ by

$$(2.1) \quad (\sigma\tau)(a) = \tau(\sigma(a)).$$

for $\sigma, \tau \in \text{Gal}(K/k)$ and $a \in K$. We denote the action of $\sigma \in \text{Gal}(K/k)$ on $x \in K$ by x^σ . This is a right action.

Let G be an arbitrary algebraic group over k . The Lie algebra of G is denoted by $\text{Lie}(G)$. We denote the identity component (the connected component containing the unit element) by G° . If G acts on a variety X and $x \in X$, then we denote the stabilizer and the orbit of x by $G_x, G(x)$ respectively.

If F is any extension field of k , then G can be regarded as an algebraic group over F . If R is a k -algebra, we denote the set of R -rational points by G_R . If K/k is a field extension, then we denote the K -algebra of regular functions in the sense of [2, p.15, 6.3] on G_K by $K[G]$. Then, $k^{\text{sep}}[G] \cong k^{\text{sep}} \otimes_k k[G]$. We define a right action of $\text{Gal}(k^{\text{sep}}/k)$ on $k^{\text{sep}}[G]$ by

$$\nu(\sigma) : k^{\text{sep}} \otimes_k k[G] \ni \sum a_i \otimes v_i \mapsto \sum a_i^\sigma \otimes v_i \in k^{\text{sep}} \otimes_k k[G].$$

For any $p \in G_{k^{\text{sep}}}$, let $p^\#$ be the associated k^{sep} -algebra homomorphism from $k^{\text{sep}}[G]$ to k^{sep} . Then $\sigma \circ p^\# \circ \nu(\sigma)^{-1}$ is also a k^{sep} -algebra homomorphism. We denote the element in $G_{k^{\text{sep}}}$ corresponding to $\sigma \circ p^\# \circ \nu(\sigma)^{-1}$ by p^σ . Then, $p^{\sigma\tau} = (p^\sigma)^\tau$ for any $\sigma, \tau \in \text{Gal}(k^{\text{sep}}/k)$. Therefore, we can define a right action of $\text{Gal}(k^{\text{sep}}/k)$ on $G_{k^{\text{sep}}}$ by

$$G_{k^{\text{sep}}} \times \text{Gal}(k^{\text{sep}}/k) \ni (p, \sigma) \mapsto p^\sigma \in G_{k^{\text{sep}}}.$$

We next define notations regarding the Galois cohomology.

Definition 2.2. For an algebraic group G over k , we consider the discrete topology on $G_{k^{\text{sep}}}$. A continuous function $h : \text{Gal}(k^{\text{sep}}/k) \rightarrow G_{k^{\text{sep}}}$ is called a 1-cocycle if $h(\sigma\tau) = h(\tau)h(\sigma)^\tau$ for any $\sigma, \tau \in \text{Gal}(k^{\text{sep}}/k)$. Two 1-cocycles h and h' are equivalent if there exists $g \in G_{k^{\text{sep}}}$ such that

$$h(\sigma) = g^{-1}h'(\sigma)g^\sigma$$

for any $\sigma \in \text{Gal}(k^{\text{sep}}/k)$. The first Galois cohomology set $H^1(k, G)$ is the set of equivalent classes of 1-cocycles by the above equivalence relation.

We shall define several notions regarding the exceptional Jordan algebra \mathcal{J} (see (1.3)) and review its fundamental properties. For the rest of this paper G, G_1, H, H_1 are the groups defined in Introduction.

Let I_n be the $n \times n$ unit matrix. We denote the diagonal matrix with diagonal entries $\alpha_1, \dots, \alpha_n$ by $\text{diag}(\alpha_1, \dots, \alpha_n)$.

Let \mathcal{O} be an octonion and \mathcal{J} the exceptional Jordan algebra as in Introduction. Let $\mathcal{O}^\times = \{x \in \mathcal{O} \mid \|x\| \neq 0\}$. This set is closed under multiplication, but may not be a group since the multiplication is not associative. However, if $x \in \mathcal{O}^\times$, then x^{-1} exists in \mathcal{O} . If $X, Y \in \mathcal{J}$, then we denote the usual matrix multiplication by XY . If $X, Y, Z \in \mathcal{J}$ and $(XY)Z = X(YZ)$, we may write XYZ .

For $X \in \mathcal{J}$, we define an endomorphism $R_X \in \text{End}(\mathcal{J})$ by

$$R_X(W) = W \circ X = \frac{1}{2}(WX + XW)$$

for $W \in \mathcal{J}$. We define a symmetric trilinear form D on \mathcal{J} by

$$\begin{aligned} 6D(X, Y, Z) = & \det(X + Y + Z) - \det(X + Y) - \det(Y + Z) - \det(Z + X) \\ & + \det(X) + \det(Y) + \det(Z). \end{aligned}$$

We denote the trace (the sum of diagonal entries) on \mathcal{J} by Tr . For $X, Y \in \mathcal{J}$, we define a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on \mathcal{J} by

$$\langle X, Y \rangle = \text{Tr}(X \circ Y).$$

One can verify by direct computation that the symmetric bilinear form $\langle \cdot, \cdot \rangle$ satisfies the following equation:

$$\langle X \circ Y, Z \rangle = \langle X, Y \circ Z \rangle, \quad \forall X, Y, Z \in \mathcal{J}.$$

For $X, Y \in \mathcal{J}$, $X \times Y$ is, by definition, the element satisfying the following equation:

$$\langle X \times Y, Z \rangle = 3D(X, Y, Z), \quad \forall Z \in \mathcal{J}.$$

Let $e = I_3$. Then the following equations are satisfied (see [12, p.122, Lemma 5.2.1]).

$$(2.3) \quad \forall X \in \mathcal{J}, \quad X \circ (X \times X) = \det(X)e, \quad e \times e = e.$$

Let

$$(2.4) \quad \begin{aligned} E_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & (\alpha)_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & \bar{\alpha} & 0 \end{pmatrix}, \\ E_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & (\beta)_2 &= \begin{pmatrix} 0 & 0 & \bar{\beta} \\ 0 & 0 & 0 \\ \beta & 0 & 0 \end{pmatrix}, \\ E_3 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & (\gamma)_3 &= \begin{pmatrix} 0 & \gamma & 0 \\ \bar{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

for $\alpha, \beta, \gamma \in \mathbb{O}$. Then,

$$(2.5) \quad \begin{aligned} (\alpha)_1 \circ (E_i) &= \begin{cases} 0 & i = 1 \\ \frac{1}{2}(\alpha)_1 & i = 2, \\ \frac{1}{2}(\alpha)_1 & i = 3 \end{cases}, \quad (\beta)_2 \circ (E_i) = \begin{cases} \frac{1}{2}(\beta)_2 & i = 1 \\ 0 & i = 2, \\ \frac{1}{2}(\beta)_2 & i = 3 \end{cases}, \\ (\gamma)_3 \circ (E_i) &= \begin{cases} \frac{1}{2}(\gamma)_3 & i = 1 \\ \frac{1}{2}(\gamma)_3 & i = 2, \\ 0 & i = 3 \end{cases}. \end{aligned}$$

We define an injective k -linear map $\iota_i : \mathbb{O} \rightarrow \mathcal{J}$ for $i = 1, 2, 3$ by

$$(2.6) \quad \iota_i(x) = (x)_i, \quad \forall x \in \mathbb{O}.$$

We denote $\iota_i(\mathbb{O}) \subset \mathcal{J}$ by \mathbb{O}_i .

The algebraic group F_4 is defined as follows:

$$(2.7) \quad F_4 = \text{Aut}(\mathcal{J}) = \{L \in G_1 \mid L(X) \circ L(Y) = L(X \circ Y), \quad \forall X, Y \in \mathcal{J}\}.$$

It is known that

$$(2.8) \quad F_4 = \{L \in G_1 \mid L(e) = e\}$$

(see [12, p.159, Proposition 5.9.4]) and F_4 is a connected simple algebraic group of type F_4 which is defined over k (see [12, p.178, Theorem 7.2.1]). Note that if $g \in G_1$ and $g(e) = e$, then by taking the determinant, $c(g) = 1$ and so $g \in H_1$.

Let \mathfrak{h}_1 and \mathfrak{f} be the Lie algebras of H_1 and F_4 respectively. Let $k[\varepsilon]/(\varepsilon^2)$ be the ring of dual numbers. Since $\text{ch}(k) \neq 3$, it is easy to see that

$$(2.9) \quad \begin{aligned} \mathfrak{h}_1 &= \{t \in \mathfrak{gl}(\mathcal{J}) \mid \det((1 + \varepsilon t)(X)) = \det(X), \quad \forall X \in \mathcal{J}\} \\ &= \{t \in \mathfrak{gl}(\mathcal{J}) \mid D(t(X), X, X) = 0, \quad \forall X \in \mathcal{J}\}, \\ \mathfrak{f} &= \{t \in \mathfrak{h}_1 \mid (1 + \varepsilon t)(e) = e\} = \{t \in \mathfrak{h}_1 \mid t(e) = 0\}. \end{aligned}$$

Moreover, it is known that $\dim \mathfrak{h}_1 = 78$ and $\dim \mathfrak{f} = 52$ (see [12, p.181, Theorem 7.3.2] and [12, p.180, Corollary 7.2.2]).

Let $\text{Der}_k(\mathcal{J}, \mathcal{J})$ be the Lie algebra of k -derivations of \mathcal{J} . The following fact is known (see [12, p.180, Corollary 7.2.2]).

Lemma 2.10. $\mathfrak{f} = \text{Der}_k(\mathcal{J}, \mathcal{J})$ and $\dim_k \text{Der}_k(\mathcal{J}, \mathcal{J}) = 52$.

We define a vector subspace $\mathfrak{D}_0 \subset \mathfrak{h}_1$ by

$$\mathfrak{D}_0 = \{X \in \text{Der}_k(\mathcal{J}, \mathcal{J}) \mid X(E_i) = 0, i = 1, 2, 3\}.$$

Let $\mathfrak{so}(Q)$ denote the Lie algebra of $\text{SO}(Q)$. Then

$$\mathfrak{so}(Q) = \{t \in \mathfrak{gl}(\mathbb{O}) \mid Q(t(x), y) + Q(x, t(y)) = 0, \quad \forall x, y \in \mathbb{O}\}.$$

If $L : \mathbb{O} \rightarrow \mathbb{O}$ is a k -linear map, then we define $\widehat{L} = \iota L \iota$ (see (1.1)). Obviously, if $L \in \text{SO}(Q)$, $\mathfrak{so}(Q)$, then $\widehat{L} \in \text{SO}(Q)$, $\mathfrak{so}(Q)$ respectively.

The following proposition is probably known (if $k = \mathbb{C}$, it is proved in [10, pp.24–27]). However, we could not find a good reference over arbitrary fields of characteristic not equal to 2, 3 and so we include the proof.

Proposition 2.11. (1) $\mathfrak{h}_1 = \text{Der}_k(\mathcal{J}, \mathcal{J}) \oplus \{R_Y \mid \text{Tr}(Y) = 0\}$.
 (2) $\text{Der}_k(\mathcal{J}, \mathcal{J}) \cong \mathfrak{D}_0 \oplus \mathfrak{O} \oplus \mathfrak{O} \oplus \mathfrak{O}$.

Proof. We shall prove this proposition in the following order.

(I) The proof of $\mathfrak{D}_0 \cong \mathfrak{so}(Q)$.

(II) The proof of (2).

(III) The proof of (1).

(I) We use the following theorem (see [12, p.53, Theorem 3.5.5] and [12, p.56, Lemma 3.5.9]).

Theorem 2.12. *If $\text{ch}(k) \neq 2$, then for any $t_1 \in \mathfrak{so}(Q)$, there exist unique $t_2, t_3 \in \mathfrak{so}(Q)$ such that*

$$(2.13) \quad t_1(xy) = t_2(x)y + xt_3(y)$$

for any $x, y \in \mathfrak{O}$. If (t_1, t_2, t_3) satisfies the equation (2.13), then (t_2, t_1, \hat{t}_3) and (t_3, \hat{t}_2, t_1) also satisfy the equation (2.13).

For any $D \in \mathfrak{D}_0$, by the equations of (2.5),

$$D((\alpha)_1 \circ (E_i)) = D((\alpha)_1) \circ E_i = \begin{cases} 0 & i = 1 \\ \frac{1}{2}D((\alpha)_1) & i = 2 \\ \frac{1}{2}D((\alpha)_1) & i = 3 \end{cases}.$$

It follows immediately that $D(\mathfrak{O}_1) \subset \mathfrak{O}_1$. Similarly, we have $D(\mathfrak{O}_i) \subset \mathfrak{O}_i$ for $i = 2, 3$. For $\alpha \in \mathfrak{O}$, we define $D_1(\alpha)$ to be the element of \mathfrak{O} such that $D((\alpha)_1) = (D_1(\alpha))_1$. We define D_2, D_3 similarly.

Since $(\alpha)_1^2 = \|\alpha\|(E_2 + E_3)$ and

$$D((\alpha)_1^2) = \|\alpha\|D(E_2 + E_3) = 0,$$

$$D((\alpha)_1^2) = 2(D_1(\alpha))_1 \circ (\alpha)_1 = 2Q(D_1(\alpha), \alpha)(E_2 + E_3),$$

we have $Q(D_1(\alpha), \alpha) = 0$. It follows that $D_1 \in \mathfrak{so}(Q)$. Similarly, we have $D_2, D_3 \in \mathfrak{so}(Q)$. We define a k -linear map r by

$$r : \mathfrak{D}_0 \ni D \mapsto \hat{D}_3 \in \mathfrak{so}(Q).$$

We prove that r is bijective.

Suppose that $t_1 \in \mathfrak{so}(Q)$. We show that t_1 is in the image of r . There exist unique t_2, t_3 satisfying (2.13) by Theorem 2.12. We define a linear map $D_{t_1} : \mathcal{J} \rightarrow \mathcal{J}$ by

$$D_{t_1} : X = \begin{pmatrix} s_1 & x_3 & \overline{x_2} \\ \overline{x_3} & s_2 & x_1 \\ x_2 & \overline{x_1} & s_3 \end{pmatrix} \mapsto \begin{pmatrix} 0 & \hat{t}_1(x_3) & \overline{t_3(x_2)} \\ \hat{t}_1(x_3) & 0 & t_2(x_1) \\ t_3(x_2) & \overline{t_2(x_1)} & 0 \end{pmatrix}.$$

By Theorem 2.12,

$$\begin{aligned}\widehat{t}_1(\overline{x_1 x_2}) &= \overline{t_2(x_1)x_2 + x_1 t_3(x_2)}, \\ t_3(\overline{x_3 x_1}) &= \overline{\widehat{t}_1(x_3)x_1 + x_3 t_2(x_1)}, \\ t_2(\overline{x_2 x_3}) &= \overline{t_3(x_2)x_3 + x_2 \widehat{t}_1(x_3)}.\end{aligned}$$

By computation,

$$X^2 = \begin{pmatrix} s_1^2 + \|x_2\| + \|x_3\| & (s_1 + s_2)x_3 + \overline{x_2 x_1} & (s_1 + s_3)\overline{x_2} + x_3 x_1 \\ (s_1 + s_2)\overline{x_3} + x_1 x_2 & s_2^2 + \|x_1\| + \|x_3\| & (s_2 + s_3)x_1 + \overline{x_3 x_2} \\ (s_1 + s_3)x_2 + \overline{x_1 x_3} & (s_2 + s_3)\overline{x_1} + x_2 x_3 & s_3^2 + \|x_1\| + \|x_2\| \end{pmatrix}.$$

Therefore,

$$\begin{aligned}D_{t_1}(X^2) &= \\ &\begin{pmatrix} 0 & (s_1 + s_2)\widehat{t}_1(x_3) + \widehat{t}_1(\overline{x_1 x_2}) & (s_1 + s_3)\overline{t_3(x_2)} + \overline{t_3(\overline{x_3 x_1})} \\ (s_1 + s_2)\widehat{t}_1(x_3) + \widehat{t}_1(\overline{x_1 x_2}) & 0 & (s_2 + s_3)t_2(x_1) + t_2(\overline{x_2 x_3}) \\ (s_1 + s_3)t_3(x_2) + t_3(\overline{x_3 x_1}) & (s_2 + s_3)\overline{t_2(x_1)} + \overline{t_2(\overline{x_2 x_3})} & 0 \end{pmatrix} \\ &= 2D_{t_1}(X) \circ X\end{aligned}$$

for any $X \in \mathcal{J}$. This implies that $D(X \circ Y) = D(X) \circ Y + X \circ D(Y)$ for all $X, Y \in \mathcal{J}$ by considering $D((X + Y) \circ (X + Y))$. Thus, $D_{t_1} \in \mathfrak{D}_0$ and so $r(D_{t_1}) = t_1$. Therefore, r is surjective.

Suppose that $D \in \mathfrak{D}_0$. Since $(\overline{\alpha\beta})_3 = 2(\alpha)_1 \circ (\beta)_2$, we have

$$\begin{aligned}D((\overline{\alpha\beta})_3) &= 2D((\alpha)_1 \circ (\beta)_2) = 2((D_1(\alpha))_1 \circ (\beta)_2 + (\alpha)_1 \circ (D_2(\beta))_2) \\ &= \left(\overline{D_1(\alpha)\beta + \alpha D_2(\beta)} \right)_3\end{aligned}$$

for any $\alpha, \beta \in \mathfrak{O}$. So $\widehat{D}_3(\alpha\beta) = D_1(\alpha)\beta + \alpha D_2(\beta)$. Therefore, $(\widehat{D}_3, D_1, D_2)$ satisfies the equation (2.13). So the uniqueness property of Theorem 2.12 implies that r is injective. Therefore, we have $\mathfrak{D}_0 \cong \mathfrak{so}(Q)$ and $\dim \mathfrak{D}_0 = 28$.

(II) We next investigate the structure of $\text{Der}_k(\mathcal{J}, \mathcal{J})$ in detail. For $\alpha, \beta, \gamma \in \mathfrak{O}$, we define elements $(\alpha)'_1, (\beta)'_2, (\gamma)'_3$ of \mathfrak{h}_1 by

$$\begin{aligned}(\alpha)'_1 &= [R_{E_2}, R_{(\alpha)_1}] (= R_{E_2}R_{(\alpha)_1} - R_{(\alpha)_1}R_{E_2}), \\ (\beta)'_2 &= [R_{E_3}, R_{(\beta)_2}], \\ (\gamma)'_3 &= [R_{E_1}, R_{(\gamma)_3}].\end{aligned}$$

We show that $(\alpha)'_1 \in \text{Der}_k(\mathcal{J}, \mathcal{J})$.

For $X \in \mathcal{J}$ as in (1.3),

$$(2.14) \quad (\alpha)'_1(X) = \frac{1}{4} \begin{pmatrix} 0 & \overline{\alpha x_2} & -x_3 \alpha \\ \alpha x_2 & 2Q(\alpha, x_1) & (-s_2 + s_3)\alpha \\ -\overline{x_3 \alpha} & (-s_2 + s_3)\overline{\alpha} & -2Q(\alpha, x_1) \end{pmatrix}.$$

Note that the following equation is satisfied (see [12, p.8, Lemma 1.3.3]).

$$(2.15) \quad (xy)\overline{z} + (xz)\overline{y} = 2Q(y, z)x, \quad \forall x, y, z \in \mathfrak{O}.$$

We have

$$\begin{aligned}
 2(\alpha)'_1(X) \circ X &= \\
 \frac{1}{4} \begin{pmatrix} 0 & (s_1 + s_3)\overline{\alpha x_2} + (x_3 x_1)\overline{\alpha} & -(s_1 + s_2)x_3\alpha - \overline{x_1 x_2}\alpha \\ (s_1 + s_3)\alpha x_2 + \alpha\overline{x_3 x_1} & 2Q((s_2 + s_3)x_1 + \overline{x_2 x_3}, \alpha) & (s_3^2 - \|x_3\| - s_2^2 + \|x_2\|)\alpha \\ -(s_1 + s_2)\overline{x_3 \alpha} - \overline{x_1 x_2}\alpha & (s_3^2 - \|x_3\| - s_2^2 + \|x_2\|)\overline{\alpha} & -2Q((s_2 + s_3)x_1 + \overline{x_2 x_3}, \alpha) \end{pmatrix} \\
 &= (\alpha)'_1(X^2)
 \end{aligned}$$

((2.15) is used for the computations of (i, j) -entries ($i \neq j$)). So, $(\alpha)'_1 \in \text{Der}_k(\mathcal{J}, \mathcal{J})$.

Similarly, we have $(\beta)'_2, (\gamma)'_3 \in \text{Der}_k(\mathcal{J}, \mathcal{J})$. Also by (2.14) and similar calculations, we have

$$\begin{aligned}
 (\alpha)'_1(E_i) &= \begin{cases} 0 & i = 1 \\ -\frac{1}{4}(\alpha)_1 & i = 2, \\ \frac{1}{4}(\alpha)_1 & i = 3 \end{cases}, \quad (\beta)'_2(E_i) = \begin{cases} \frac{1}{4}(\beta)_2 & i = 1 \\ 0 & i = 2, \\ -\frac{1}{4}(\beta)_2 & i = 3 \end{cases}, \\
 (\gamma)'_3(E_i) &= \begin{cases} -\frac{1}{4}(\gamma)_3 & i = 1 \\ \frac{1}{4}(\gamma)_3 & i = 2, \\ 0 & i = 3 \end{cases}.
 \end{aligned}
 \tag{2.16}$$

We define a k -linear map l by

$$\begin{aligned}
 l: \underbrace{\mathfrak{D}_0 \oplus \mathbf{O} \oplus \mathbf{O} \oplus \mathbf{O}}_{\Psi} &\rightarrow \underbrace{\text{Der}_k(\mathcal{J}, \mathcal{J})}_{\Psi} \\
 (X_0, \alpha, \beta, \gamma) &\mapsto X_0 + (\alpha)'_1 + (\beta)'_2 + (\gamma)'_3.
 \end{aligned}$$

We show that l is bijective. For that purpose, it is enough to prove that l is surjective since the dimensions of both sides are equal to 52 (see Lemma 2.10 and (I)).

For $D \in \text{Der}_k(\mathcal{J}, \mathcal{J})$, let $X_i = D(E_i) = h(s_{i1}, s_{i2}, s_{i3}, x_{i1}, x_{i2}, x_{i3}) \in \mathcal{J}$ where $i = 1, 2, 3$ (see (1.3)). Then,

$$D(E_i) = D(E_i^2) = 2E_i \circ D(E_i).$$

We consider the case $i = 1$. Since

$$X_1 = D(E_1) = 2E_1 \circ D(E_1) = \begin{pmatrix} 2s_{11} & x_{13} & \overline{x_{12}} \\ \overline{x_{13}} & 0 & 0 \\ x_{12} & 0 & 0 \end{pmatrix},$$

we have $s_{11} = s_{12} = s_{13} = x_{11} = 0$. Therefore, $X_1 = (x_{12})_2 + (x_{13})_3$. By similar calculations, we have $s_{2i} = s_{3i} = 0$ for $i = 1, 2, 3$, $x_{22} = x_{33} = 0$ and $X_2 = (x_{23})_3 + (x_{21})_1$, $X_3 = (x_{31})_1 + (x_{32})_2$. Moreover, since

$$0 = D(E_i \circ E_j) = D(E_i) \circ E_j + E_i \circ D(E_j) = X_i \circ E_j + E_i \circ X_j$$

for any $i \neq j \in \{1, 2, 3\}$,

$$(2.17) \quad \begin{aligned} (x_{13})_3 + (x_{23})_3 &= 0 & \text{from } (i, j) &= (1, 2), \\ (x_{12})_2 + (x_{32})_2 &= 0 & \text{from } (i, j) &= (1, 3), \\ (x_{21})_1 + (x_{31})_1 &= 0 & \text{from } (i, j) &= (2, 3). \end{aligned}$$

Hence, using (2.16), (2.17), we have

$$(D - 4(x_{31})'_1 - 4(x_{12})'_2 - 4(x_{23})'_3)(E_i) = 0.$$

Let

$$D_0 = D - 4(x_{31})'_1 - 4(x_{12})'_2 - 4(x_{23})'_3.$$

Then $D_0 \in \mathfrak{D}_0$ and

$$l((D_0, 4x_{31}, 4x_{12}, 4x_{23})) = D.$$

Therefore, l is surjective. So we conclude that $\text{Der}_k(\mathcal{J}, \mathcal{J}) \cong \mathfrak{D}_0 \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$.

(III) Finally, we prove (1). For any R_Y where $\text{Tr}(Y) = \langle Y, e \rangle = 0$,

$$\begin{aligned} D(R_Y(X), X, X) &= \frac{1}{3} \langle Y \circ X, X \times X \rangle = \frac{1}{3} \langle Y, X \circ (X \times X) \rangle \\ &= \frac{1}{3} \det(X) \langle Y, e \rangle = 0 \end{aligned}$$

for any $X \in \mathcal{J}$. Thus, $R_Y \in \mathfrak{h}_1$.

For $t \in \mathfrak{h}_1$, let $Y = t(e)$. Then, since $D(t(e), e, e) = 0$, by (2.3), we have

$$D(t(e), e, e) = \frac{1}{3} \langle t(e), e \times e \rangle = \frac{1}{3} \langle Y, e \rangle = 0,$$

It follows that $\text{Tr}(Y) = 0$. Since $(t - R_Y)(e) = 0$, by (2.9), $t - R_Y \in \text{Der}_k(\mathcal{J}, \mathcal{J})$. Thus,

$$\mathfrak{h}_1 = \text{Der}_k(\mathcal{J}, \mathcal{J}) + \{R_Y \mid \text{Tr}(Y) = 0\}.$$

Since $\dim \mathfrak{h}_1 = 78 = \dim \text{Der}_k(\mathcal{J}, \mathcal{J}) + \dim \{R_Y \mid \text{Tr}(Y) = 0\}$, the above sum is a direct sum. This completes the proof of (1). \square

We define varieties M and $RT(\mathcal{O})$ by

$$M = \{(A, B, C) \in \text{SO}(Q) \times \text{SO}(Q) \times \text{SO}(Q) \mid A(x)B(y) = \widehat{C}(xy), \forall x, y \in \mathcal{O}\},$$

$$RT(\mathcal{O}) = \{(A, B, C) \in \text{SO}(Q) \times \text{SO}(Q) \times \text{SO}(Q) \mid A(xy) = B(x)C(y), \forall x, y \in \mathcal{O}\}$$

respectively. Then M and $RT(\mathcal{O})$ are defined over k .

The variety $RT(\mathcal{O})$ is given in [12, p.59]. It is known that $RT(\mathcal{O})$ is a closed subgroup of the algebraic group $\text{SO}(Q) \times \text{SO}(Q) \times \text{SO}(Q)$ (see [12, p.59]). We show that M is also a subgroup of $\text{SO}(Q) \times \text{SO}(Q) \times \text{SO}(Q)$ and is isomorphic to $RT(\mathcal{O})$ over k .

For any $(A, B, C), (A', B', C') \in M$ and $x, y \in \mathcal{O}$, we have

$$\begin{aligned} (A'A)(x)(B'B)(y) &= A'(A(x))B'(B(y)) = \widehat{C}'(A(x)B(y)) \\ &= (\widehat{C}'\widehat{C})(xy) = (\widehat{C'}\widehat{C})(xy), \\ (A^{-1})(x)(B^{-1})(y) &= (\widehat{C}^{-1}\widehat{C})(A^{-1}(x)B^{-1}(y)) \\ &= \widehat{C}^{-1}(A(A^{-1}(x))(B(B^{-1}(y)))) \\ &= \widehat{C}^{-1}(xy) = \widehat{C^{-1}}(xy). \end{aligned}$$

It follows that M is closed under multiplication and taking the inverse. Obviously, the unit element (I_8, I_8, I_8) of $\mathrm{SO}(Q) \times \mathrm{SO}(Q) \times \mathrm{SO}(Q)$ belongs to M . Therefore, M is a subgroup of $\mathrm{SO}(Q) \times \mathrm{SO}(Q) \times \mathrm{SO}(Q)$.

We define a homomorphism f by

$$f : M \ni (A, B, C) \mapsto (\widehat{C}, A, B) \in \mathrm{RT}(\mathcal{O}).$$

Then, f is obviously an isomorphism defined over k . So M is isomorphic to $\mathrm{RT}(\mathcal{O})$ over k . Since $\mathrm{RT}(\mathcal{O})$ is smooth over k (see [12, p.60]), M is also smooth over k .

It is known that the algebraic group $\mathrm{RT}(\mathcal{O})$ satisfies the following properties. Note that we are assuming that $\mathrm{ch}(k) \neq 2$.

Proposition 2.18.

- (i) Let $\mathrm{pr}_i : \mathrm{RT}(\mathcal{O}) \rightarrow \mathrm{SO}(Q)$ be the projection $\mathrm{pr}_i((t_1, t_2, t_3)) = t_i$ ($i = 1, 2, 3$). Then pr_1 is a surjective homomorphism from $\mathrm{RT}(\mathcal{O})$ to $\mathrm{SO}(Q)$ with kernel of order 2 and the representations pr_i on \mathcal{O} are irreducible and pairwise inequivalent.
- (ii) Let η_1, η_2, η_3 be the following automorphisms of $\mathrm{RT}(\mathcal{O})$

$$\begin{aligned} \eta_1 &: (t_1, t_2, t_3) \rightarrow (\widehat{t}_1, \widehat{t}_3, \widehat{t}_2) \\ \eta_2 &: (t_1, t_2, t_3) \rightarrow (t_3, \widehat{t}_2, t_1) . \\ \eta_3 &: (t_1, t_2, t_3) \rightarrow (t_2, t_1, \widehat{t}_3) \end{aligned}$$

Then $\eta_i^2 = 1$ ($i = 1, 2, 3$) and $\{\eta_1, \eta_2, \eta_3\}$ generates a group \tilde{S} of outer automorphisms of $\mathrm{RT}(\mathcal{O})$ which is isomorphic to \mathfrak{S}_3 .

- (iii) $\mathrm{RT}(\mathcal{O})$ is isomorphic to $\mathrm{Spin}(Q)$ as algebraic groups.

Proof. See [12, p.59, Proposition 3.6.1], [12, p.60, Proposition 3.6.3] and [12, p.64, Proposition 3.6.6,]. \square

For the definition of the algebraic group $\mathrm{Spin}(Q)$, see [12, pp.38,39]. It is known that $\mathrm{Spin}(Q)$ is connected (see [12, p.40]). Moreover, there exists a surjective homomorphism from $\mathrm{Spin}(Q)$ to $\mathrm{SO}(Q)$ with kernel of order 2 if $\mathrm{ch}(k) \neq 2$ (see [12, p.40]). Since $\mathrm{SO}(Q)$ is semi-simple, by [2, p.192, 14.11 Corollary], $\mathrm{Spin}(Q)$ is semi-simple. So $\mathrm{RT}(\mathcal{O})$ is also connected and semi-simple.

Since M is isomorphic to $\mathrm{RT}(\mathcal{O})$, we have the following proposition (we are still assuming that $\mathrm{ch}(k) \neq 2$).

Proposition 2.19.

- (i) Let $\mathrm{pr}_i : M \rightarrow \mathrm{SO}(Q)$ be the projection $\mathrm{pr}_i((t_1, t_2, t_3)) = t_i$ ($i = 1, 2, 3$). Then pr_3 is a surjective homomorphism from M to $\mathrm{SO}(Q)$ with kernel of order 2 and the representations pr_i on \mathcal{O} are irreducible and pairwise inequivalent.

(ii) Let $\eta'_1, \eta'_2, \eta'_3$ be the following automorphisms of M

$$\begin{aligned}\eta'_1 &: (t_1, t_2, t_3) \rightarrow (\widehat{t_1}, \widehat{t_3}, \widehat{t_2}) \\ \eta'_2 &: (t_1, t_2, t_3) \rightarrow (\widehat{t_3}, \widehat{t_2}, \widehat{t_1}) . \\ \eta'_3 &: (t_1, t_2, t_3) \rightarrow (\widehat{t_2}, \widehat{t_1}, \widehat{t_3})\end{aligned}$$

Then $\eta_i'^2 = 1$ ($i = 1, 2, 3$) and $\{\eta'_1, \eta'_2, \eta'_3\}$ generates a group of outer automorphisms \tilde{S}' of M which is isomorphic to \mathfrak{S}_3 .

(iii) M is isomorphic to $\text{Spin}(Q)$ as algebraic groups.

Note that in the above proposition, $\eta'_1, \eta'_2, \eta'_3$ correspond to η_2, η_3, η_1 respectively. For example, by the isomorphism $M \cong RT(\mathcal{O})$ and η_1 , we have a map

$$(t_1, t_2, t_3) \rightarrow (\widehat{t_3}, t_1, t_2) \rightarrow (t_3, \widehat{t_2}, \widehat{t_1}) \rightarrow (\widehat{t_2}, \widehat{t_1}, \widehat{t_3}),$$

which is η'_3 .

From now on, we denote M by $\text{Spin}(Q)$.

3. STABILIZER

In this section, we shall determine the structure of the stabilizer G_w as an algebraic group.

Theorem 3.1. G_w is isomorphic to $\text{GL}(1) \times (\text{Spin}(Q) \rtimes \mathfrak{S}_3)$. In particular, G_w is a smooth reductive algebraic group.

Proof. To prove this proposition, we first determine the identity component G_w° of the stabilizer mainly by Lie algebra computations. Next, given an arbitrary element $g \in G_w$, we replace g by the action of $\text{Spin}(Q) \rtimes \mathfrak{S}_3$ so that it commutes with all elements of $\text{Spin}(Q)$. Then Schur's lemma enables us to simplify the situation enough to prove the proposition.

We first prove the following proposition.

Proposition 3.2. (1) G_w° is isomorphic to $\text{GL}(1) \times \text{Spin}(Q)$ over k .
 (2) (G, V) is an irreducible prehomogeneous vector space.

Proof. We shall prove this proposition in the following two steps.

(I) We show that $\dim G_w = 29$ and G_w is smooth. In this process, we prove (2).

(II) We construct an injective homomorphism $\xi : \text{GL}(1) \times \text{Spin}(Q) \rightarrow G_w$ which is defined over k and induces an isomorphism between $\text{Lie}(\text{GL}(1) \times \text{Spin}(Q))$ and $\text{Lie}(G_w)$.

If (I) and (II) are proved, since the dimensions of both $\text{GL}(1) \times \text{Spin}(Q)$ and G_w are equal to 29 and both algebraic groups are smooth, we have $\text{GL}(1) \times \text{Spin}(Q) \cong G_w^\circ$ by ξ .

(I) Let \mathfrak{g}_w and \mathfrak{h}_w denote the Lie algebras of G_w and H_w respectively. We shall prove that $\mathfrak{h}_w = \mathfrak{D}_0$ and $\dim H_w = 28$. Since obviously $\mathfrak{D}_0 \subset \mathfrak{h}_w$, it is enough to show that $\mathfrak{h}_w \subset \mathfrak{D}_0$.

Let $Z = (X, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in \mathfrak{h}_w \subset \mathfrak{h}_1 \oplus \mathfrak{gl}(2)$, where $X \in \mathfrak{h}_1$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{gl}(2)$. Then by Proposition 2.11, there exists $X_0 \in \mathfrak{D}_0$, $\alpha, \beta, \gamma \in \mathcal{O}$ and $Y \in \mathcal{J}$ such that

$$X = X_0 + (\alpha)'_1 + (\beta)'_2 + (\gamma)'_3 + R_Y$$

and $\text{Tr}(Y) = 0$.

Let $T = (\alpha)'_1 + (\beta)'_2 + (\gamma)'_3$. Then

$$(3.3) \quad T \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{(\alpha)_1}{4} + \frac{(\beta)_2}{4} - \frac{(\gamma)_3}{2},$$

$$T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -\frac{(\alpha)_1}{2} + \frac{(\beta)_2}{4} + \frac{(\gamma)_3}{4}.$$

Let $Y = \begin{pmatrix} t_1 & y_3 & \overline{y_2} \\ \overline{y_3} & t_2 & y_1 \\ y_2 & \overline{y_1} & t_3 \end{pmatrix}$, where $t_i \in k$, $y_i \in \mathbb{O}$ ($i = 1, 2, 3$) and $t_1 + t_2 + t_3 = 0$.

Then

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \circ Y &= \frac{1}{2} \left(\begin{pmatrix} t_1 & y_3 & \overline{y_2} \\ -\overline{y_3} & -t_2 & -y_1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} t_1 & -y_3 & 0 \\ \overline{y_3} & -t_2 & 0 \\ y_2 & -\overline{y_1} & 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} t_1 & 0 & \frac{1}{2}\overline{y_2} \\ 0 & -t_2 & -\frac{1}{2}y_1 \\ \frac{1}{2}y_2 & -\frac{1}{2}\overline{y_1} & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \circ Y &= \begin{pmatrix} 0 & \frac{1}{2}y_3 & -\frac{1}{2}\overline{y_2} \\ \frac{1}{2}\overline{y_3} & t_2 & 0 \\ -\frac{1}{2}y_2 & 0 & -t_3 \end{pmatrix}. \end{aligned}$$

Since $Zw = (X, \begin{pmatrix} a & b \\ c & d \end{pmatrix})w = 0$, we have

$$(3.4) \quad T \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} a & 0 & 0 \\ 0 & -a+b & 0 \\ 0 & 0 & -b \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \circ Y = 0$$

and

$$(3.5) \quad T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} c & 0 & 0 \\ 0 & -c+d & 0 \\ 0 & 0 & -d \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \circ Y = 0.$$

By (3.3), (3.4) and (3.5), we have

$$(3.6) \quad \begin{pmatrix} a+t_1 & -\frac{1}{2}\gamma & \frac{1}{2}\overline{y_2} + \frac{1}{4}\overline{\beta} \\ -\frac{1}{2}\overline{\gamma} & -a+b-t_2 & -\frac{1}{2}y_1 + \frac{1}{4}\alpha \\ \frac{1}{2}y_2 + \frac{1}{4}\beta & -\frac{1}{2}\overline{y_1} + \frac{1}{4}\overline{\alpha} & -b \end{pmatrix} = 0$$

and

$$(3.7) \quad \begin{pmatrix} c & \frac{1}{2}y_3 + \frac{1}{4}\gamma & -\frac{1}{2}\overline{y_2} + \frac{1}{4}\overline{\beta} \\ \frac{1}{2}\overline{y_3} + \frac{1}{4}\overline{\gamma} & -c+d+t_2 & -\frac{1}{2}\alpha \\ -\frac{1}{2}y_2 + \frac{1}{4}\beta & -\frac{1}{2}\overline{\alpha} & -d-t_3 \end{pmatrix} = 0.$$

By (3.6), we have

$$b = 0, t_1 = -a, t_2 = -a, \gamma = 0, y_1 = \frac{\alpha}{2}, y_2 = -\frac{\beta}{2}.$$

By (3.7), we have

$$c = 0, t_2 = -d, t_3 = -d, \alpha = 0, y_3 = 0, y_2 = \frac{\beta}{2}.$$

Hence, we have

$$\begin{aligned} t_1 &= t_2 = t_3 = -a = -d, \\ y_1 &= y_2 = y_3 = \alpha = \beta = \gamma = b = c = 0. \end{aligned}$$

Since $\text{Tr}(Y) = 0$, $3t_1 = 0$ ($\text{ch}(k) \neq 3$). Therefore, $Y = 0$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$ and $\alpha = \beta = \gamma = 0$. Thus, $Z = X_0$, and so $\mathfrak{h}_w = \mathfrak{D}_0$. Then, we also have $\dim H_w \leq \dim \mathfrak{h}_w = \dim \mathfrak{D}_0 = 28$.

Since $\dim H_w + \dim H(w) = \dim H$ ($H(w)$ is the orbit), we have

$$\dim H_w = \dim H - \dim H(w) \geq \dim H - \dim V \geq 82 - 54 = 28.$$

It follows that $\dim H_w = 28$. Since $\dim H = 82$ and $H_w = 28$, $\dim H(w) = 54$. Moreover, it is a dense subset of V .

We show that $\dim G_w = 29$. Since $\dim H_1 = \dim \mathfrak{h}_1 = 78$, by the exact sequence (1.6), we have $\dim G_1 = 79$. Thus, since the dimension of the G -orbit of w is also 54 and $\dim G = 83$, we have $\dim G_w = 29$. Since $G(w)$ is a constructible subset which is irreducible, it is locally closed. Moreover, since $\dim G(w) = 54 = \dim V$, $G(w)$ is open in V . It follows that the condition of Definition 1.9 (i) is satisfied. We pointed out in Introduction that V is an irreducible representation and that there exists a relative invariant polynomial. So we have (2) of the proposition.

We prove that G_w is smooth. For that purpose, it is enough to show that $\dim \mathfrak{g}_w = \dim G_w = 29$. Let \mathfrak{g}_1 denote the Lie algebra of G_1 . We define dc and dc' by the Lie algebra homomorphisms induced by the characters $c : G_1 \rightarrow \text{GL}(1)$ and $c' : G \rightarrow \text{GL}(1)$ respectively (see (1.8)). Then,

$$\mathfrak{g}_1 = \left\{ L \in \mathfrak{gl}(\mathcal{J}) \mid \begin{array}{l} \forall X \in \mathcal{J}, \det((1 + \varepsilon L)(X)) = c(1 + \varepsilon L) \det(X) \\ \phantom{\forall X \in \mathcal{J},} = (1 + \varepsilon dc(L)) \det(X) \end{array} \right\}.$$

For $a \in k$,

$$\det((1 + a\varepsilon)I_{27}(X)) = (1 + a\varepsilon)^3 \det(X) = (1 + 3a\varepsilon) \det(X).$$

So $dc(aI_{27}) = 3a$. Since $\text{ch}(k) \neq 3$, $dc : \mathfrak{g}_1 \rightarrow \mathfrak{gl}(1)$ is surjective. Hence, the sequence

$$0 \rightarrow \mathfrak{h}_1 \rightarrow \mathfrak{g}_1 \rightarrow \mathfrak{gl}(1) \rightarrow 0$$

is exact. For $L = (L_1, L_2) \in \mathfrak{g}_w \subset \mathfrak{g}_1 \oplus \mathfrak{gl}(2)$, if $dc'(L) = 0$, then $dc(L_1) = 0$. It follows that $L_1 \in \mathfrak{h}_1$ and $L \in \mathfrak{h}_w$. Since $dc'((aI_{27}, -aI_2)) = 3a \in \mathfrak{gl}(1)$, the restriction of dc' on \mathfrak{g}_w is also surjective. Hence, the following sequence

$$0 \rightarrow \mathfrak{h}_w \rightarrow \mathfrak{g}_w \rightarrow \mathfrak{gl}(1) \rightarrow 0$$

is exact. Therefore, $\dim \mathfrak{g}_w = 29$ and G_w is smooth.

(II) We shall construct an injective homomorphism $\zeta : \text{GL}(1) \times \text{Spin}(Q) \rightarrow G_w$.

We identify \mathcal{J} with $k^3 \oplus \mathbf{O} \oplus \mathbf{O} \oplus \mathbf{O}$ by the following isomorphism:

$$(3.8) \quad \begin{array}{ccc} \mathcal{J} & \cong & k^3 \oplus \mathbf{O} \oplus \mathbf{O} \oplus \mathbf{O} \\ \cup & & \cup \\ \begin{pmatrix} s_1 & x_3 & \overline{x_2} \\ \overline{x_3} & s_2 & x_1 \\ x_2 & \overline{x_1} & s_3 \end{pmatrix} & \leftrightarrow & ((s_1, s_2, s_3), x_1, x_2, x_3) \end{array}.$$

We choose $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ as the basis of k^3 . By the isomorphism $\mathcal{J} \cong k^3 \oplus \mathbf{O} \oplus \mathbf{O} \oplus \mathbf{O}$, we express elements of $GL(\mathcal{J})$ in 4×4 block form.

We define a homomorphism $\xi_0 : GL(1) \times Spin(Q) \rightarrow GL(\mathcal{J})$ by

$$(3.9) \quad \xi_0((t, (A, B, C))) = \begin{pmatrix} tI_3 & 0 & 0 & 0 \\ 0 & tA & 0 & 0 \\ 0 & 0 & tB & 0 \\ 0 & 0 & 0 & tC \end{pmatrix}$$

for $(t, A, B, C) \in GL(1) \times Spin(Q)$.

Let $Z = (t, (A, B, C)) \in GL(1) \times Spin(Q)$. Note that we are identifying $Spin(Q)$ with M . We prove that $\xi_0(Z) \in G_1$. For any X in the form (1.3), we have

$$\xi_0(Z)X = \begin{pmatrix} ts_1 & tC(x_3) & \overline{tB(x_2)} \\ t\overline{C(x_3)} & ts_2 & tA(x_1) \\ tB(x_2) & \overline{tA(x_1)} & ts_3 \end{pmatrix}$$

and

$$\begin{aligned} \det(\xi_0(Z)X) &= t^3 s_1 s_2 s_3 + \text{tr}(tA(x_1)tB(x_2)tC(x_3)) \\ &\quad - ts_1 \|tA(x_1)\| - ts_2 \|tB(x_2)\| - ts_3 \|tC(x_3)\|. \end{aligned}$$

Since $A, B, C \in SO(Q)$, they preserve the inner product. Thus, we have $\|tA(x_1)\| = t^2 \|x_1\|$, $\|tB(x_2)\| = t^2 \|x_2\|$, $\|tC(x_3)\| = t^2 \|x_3\|$ and

$$\begin{aligned} \text{tr}(tA(x_1)tB(x_2)tC(x_3)) &= t^3 \text{tr}\left(\overline{C(\overline{x_1 x_2})} C(x_3)\right) = 2t^3 Q(\overline{x_1 x_2}, x_3) \\ &= t^3 \text{tr}(x_1 x_2 x_3). \end{aligned}$$

Therefore, $\det(\xi_0(Z)X) = t^3 \det(X)$, and so $\xi_0(Z) \in G_1$.

We define a homomorphism ξ by

$$\xi : GL(1) \times Spin(Q) \ni Z = (t, (A, B, C)) \mapsto (\xi_0(Z), t^{-1}I_2) \in G_1 \times GL(2).$$

By (3.9), $\text{Im}(\xi) \subset G_w$. Since $GL(1) \times Spin(Q)$ is connected, we have $\text{Im}(\xi) \subset G_w^\circ$. Moreover, ξ is obviously injective and defined over k . Thus, since $\dim GL(1) \times Spin(Q) = 29$, we have $\dim \xi(GL(1) \times Spin(Q)) = 29$.

The induced homomorphism of Lie algebras from $\text{Lie}(\text{GL}(1) \times \text{Spin}(Q))$ to \mathfrak{g}_w is as follows:

$$\begin{array}{ccc}
 \mathfrak{gl}(1) \oplus \mathfrak{so}(Q) \oplus \mathfrak{so}(Q) \oplus \mathfrak{so}(Q) & & \mathfrak{g}_1 \oplus \mathfrak{gl}(2) \\
 \cup & & \cup \\
 d\tilde{\zeta} : \text{Lie}(\text{GL}(1) \times \text{Spin}(Q)) & \longrightarrow & \mathfrak{g}_w \\
 \cup & & \cup \\
 (t, U_1, U_2, U_3) & \longmapsto & \left(tI_{27} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & U_1 & 0 & 0 \\ 0 & 0 & U_2 & 0 \\ 0 & 0 & 0 & U_3 \end{pmatrix}, \begin{pmatrix} -t & 0 \\ 0 & -t \end{pmatrix} \right)
 \end{array}$$

Note that (\hat{U}_3, U_1, U_2) satisfies the equation (2.13). Since $d\tilde{\zeta}$ is injective and $\dim \text{Lie}(\text{GL}(1) \times \text{Spin}(Q)) = \dim \mathfrak{g}_w = 29$, $d\tilde{\zeta}$ is an isomorphism. Thus, $G_w^\circ \cong \text{GL}(1) \times \text{Spin}(Q)$ over k . This completes the proof of Proposition 3.2 \square

From now on, we identify G_w° with $\text{GL}(1) \times \text{Spin}(Q)$ by the above isomorphism.

We next determine the structure of G_w/G_w° . We prove that $G_w/G_w^\circ \cong \mathfrak{S}_3$. Since G_w and G_w° are defined over k , G_w/G_w° and the natural homomorphism $\pi : G_w \rightarrow G_w/G_w^\circ$ are defined over k (see [?, p.]8, 6.8 Theorem]Borel). We prove that there is a finite subgroup of G_w which is isomorphic to \mathfrak{S}_3 and is mapped bijectively to G_w/G_w° by π .

Let

$$\begin{aligned}
 \tau_1 &= \left(\left(\begin{array}{ccc|ccc} 0 & 1 & 0 & & & \\ 1 & 0 & 0 & & & \\ 0 & 0 & 1 & & & \\ \hline & & & 0 & & \\ 0 & & & 0 & \iota & 0 \\ & & & \iota & 0 & 0 \\ & & & 0 & 0 & \iota \end{array} \right), \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \right), \\
 \tau_2 &= \left(\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & 0 & 1 & & & \\ 0 & 1 & 0 & & & \\ \hline & & & 0 & & \\ \iota & & & 0 & 0 & 0 \\ & & & 0 & 0 & \iota \\ & & & 0 & \iota & 0 \end{array} \right), \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \right), \\
 e_G &= (I_{27}, I_2)
 \end{aligned}
 \tag{3.10}$$

(see (1.1)). Then, by easy computations, we have $\tau_1^2 = \tau_2^2 = e_G$, $(\tau_1 \tau_2)^2 = \tau_2 \tau_1$ and $\tau_1, \tau_2 \in G_w$. Let $\langle \tau_1, \tau_2 \rangle$ denote the finite subgroup of G_w generated by τ_1, τ_2 . It is easy to see that $|\langle \tau_1, \tau_2 \rangle| = 6$.

For $g \in G_w$, we define $I_g \in \text{Aut}(G_w)$ by

$$I_g : G_w \ni x \mapsto gxg^{-1} \in G_w.$$

Since G_w° is a normal subgroup of G_w , by restricting I_g to G_w° , I_g induces an element of $\text{Aut}(G_w^\circ)$. Moreover, since $\text{GL}(1)$ is contained in the center of G_w , I_g also induces an element of $\text{Aut}(\text{Spin}(Q))$. So, we regard $I_g \in \text{Aut}(\text{Spin}(Q))$. Then, the map

$$G_w \ni g \mapsto I_g \in \text{Aut}(\text{Spin}(Q))$$

is a homomorphism. We denote this homomorphism by I .

We define a subgroup $\text{Inn}(\text{Spin}(Q))$ of $\text{Aut}(\text{Spin}(Q))$ by

$$\text{Inn}(\text{Spin}(Q)) = \{I_h \in \text{Aut}(\text{Spin}(Q)) \mid h \in \text{Spin}(Q)\}.$$

Since $\text{Spin}(Q)$ is a group of type D_4 and the automorphism group of the Dynkin diagram of D_4 is \mathfrak{S}_3 , by Proposition 2.19(ii) and [2, p.190, Proposition], the following sequence

$$(3.11) \quad 0 \rightarrow \text{Inn}(\text{Spin}(Q)) \rightarrow \text{Aut}(\text{Spin}(Q)) \rightarrow \mathfrak{S}_3 \rightarrow 0$$

is exact. We denote the above homomorphism from $\text{Aut}(\text{Spin}(Q))$ to \mathfrak{S}_3 by f . The definition of f is as follows. For $Z = (Z_1, Z_2, Z_3) \in \text{Spin}(Q)$, we define $\text{pr}_i(Z) = Z_i$ ($i = 1, 2, 3$). For any $x \in \text{Aut}(\text{Spin}(Q))$ and $i \in \{1, 2, 3\}$, $\text{pr}_i \circ x$ is an irreducible representation of $\text{Spin}(Q)$ on \mathcal{O} and equivalent to some pr_j where $j \in \{1, 2, 3\}$. Then we define $f(x)(i) = j$.

We denote the composition of homomorphisms

$$G_w \xrightarrow{I} \text{Aut}(\text{Spin}(Q)) \xrightarrow{f} \mathfrak{S}_3$$

by φ . It is easy to see that the representation $Z = (Z_1, Z_2, Z_3) \mapsto \widehat{Z}_i$ is equivalent to pr_i for $i = 1, 2, 3$. Since $I_{\tau_1}(Z) = (\widehat{Z}_2, \widehat{Z}_1, \widehat{Z}_3)$, $I_{\tau_2}(Z) = (\widehat{Z}_1, \widehat{Z}_3, \widehat{Z}_2)$, we have

$$(3.12) \quad \varphi(\tau_1) = (1\ 2), \quad \varphi(\tau_2) = (2\ 3).$$

It follows that $\varphi(\langle \tau_1, \tau_2 \rangle) = \mathfrak{S}_3$. Since $|\langle \tau_1, \tau_2 \rangle| = 6$, $\langle \tau_1, \tau_2 \rangle \cong \mathfrak{S}_3$ and $\varphi : G_w \rightarrow \mathfrak{S}_3$ is surjective.

We define another natural homomorphism from G_w to \mathfrak{S}_3 as follows. For $x = x_1v_1 + x_2v_2 \in V$, let F_x be the cubic form defined in (1.11). For $x \in V$, we define

$$(3.13) \quad \text{Zero}(x) = \{q \in \mathbb{P}_k^1 \mid F_x(q) = 0\}.$$

We call $\text{Zero}(x)$ the *zero set* of x . Note that this is well-defined since F_x is homogeneous. We express elements of \mathbb{P}_k^1 by row vectors as $q = (q_1, q_2)$.

We shall define an action of G_x on $\text{Zero}(x)$. For any $g = (g_1, g_2) \in G_x$,

$$F_x(v) = F_{gx}(v) = c(g_1)F_x(vg_2).$$

It follows that if $q \in \text{Zero}(x)$, then qg_2 also belongs to $\text{Zero}(x)$. Hence, we can define a right action of G_x on $\text{Zero}(x)$ by

$$G_x \times \text{Zero}(x) \ni ((g_1, g_2), q) \mapsto qg_2 \in \text{Zero}(x).$$

Let $\text{Zero}(x) = \{q_{x,1}, q_{x,2}, q_{x,3}\}$. For $g = (g_1, g_2) \in G_x$, let $\eta_x(g) \in \mathfrak{S}_3$ be the element such that $q_{x,i}g_2 = q_{x,\eta_x(g)(i)}$ for $i = 1, 2, 3$.

We define a map η_x by

$$(3.14) \quad \eta_x : G_x \ni g \mapsto \eta_x(g) \in \mathfrak{S}_3.$$

For any $q_{x,i} \in \text{Zero}(x)$, $g = (g_1, g_2)$ and $h = (h_1, h_2) \in G_x$,

$$(3.15) \quad q_{x,\eta(g)(i)} = q_{x,i}g_2h_2 = q_{x,\eta(g)(i)}h_2 = q_{x,\eta(h)(\eta(g)(i))}.$$

Thus,

$$\eta_x(gh)(i) = \eta_x(h)(\eta_x(g)(i)).$$

Therefore, the map $\eta_x : G_x \rightarrow \mathfrak{S}_3$ is a homomorphism. The definition of η_x depends on the order of $\{q_{x,1}, q_{x,2}, q_{x,3}\}$ but its conjugacy class depends only on x .

We remind the reader that $w \in V_k$ is the element defined in (1.13). By direct computation, we have $F_w(v_1, v_2) = v_1 v_2 (v_1 - v_2)$. It follows that

$$\text{Zero}(w) = \{(0, 1), (1, 1), (1, 0)\} \subset \mathbb{P}_1^k.$$

By choosing $q_{w,1} = (0, 1), q_{w,2} = (1, 1), q_{w,3} = (1, 0)$, we obtain a homomorphism $\eta_w : G_w \rightarrow \mathfrak{S}_3$. We shall show at the end of this section that the following diagram is commutative.

$$(3.16) \quad \begin{array}{ccc} G_w & \xrightarrow{\eta_w} & \mathfrak{S}_3 \\ I \downarrow & \circlearrowright & \nearrow f \\ \text{Aut}(\text{Spin}(Q)) & & \end{array}$$

Next, we show that the following sequence

$$(3.17) \quad 0 \rightarrow G_w^\circ \rightarrow G_w \xrightarrow{\varphi} \mathfrak{S}_3 \rightarrow 0$$

is exact and split. We may assume that k is algebraically closed. Since φ is surjective, we only have to show that $G_w^\circ = \text{Ker}(\varphi)$. Since G_w° is connected, $G_w^\circ \subset \text{Ker}(\varphi)$. So we shall show that $G_w^\circ \supset \text{Ker}(\varphi)$.

Let $g = (g_1, g_2) \in \text{Ker}(\varphi) \subset G_1 \times \text{GL}(2)$. Since $g \in \text{Ker}(\varphi)$, $f(I_g) = 1$. Hence, by the exact sequence (3.11), $I_g \in \text{Inn}(\text{Spin}(Q))$. It follows that there exists $h \in \text{Spin}(Q)$ such that I_{gh} is trivial on $\text{Spin}(Q)$ (since $k = \bar{k}$). Thus, we may assume that I_g is trivial on $\text{Spin}(Q)$.

Since g commutes with all elements of $\text{Spin}(Q)$, g_1 preserves each irreducible non-equivalent representation of $\text{Spin}(Q)$ in \mathcal{J} . Thus, g_1 is in the block form:

$$g_1 = \begin{pmatrix} A_0 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_3 \end{pmatrix},$$

where $A_0 \in \text{GL}(3), A_1, A_2, A_3 \in \text{GL}(\mathcal{O})$. Moreover, since $A_i Z_i = Z_i A_i$ for all $i \in \{1, 2, 3\}$ and $Z = (Z_1, Z_2, Z_3) \in \text{Spin}(Q)$, by Schur's lemma [3, p.7] ($k = \bar{k}$), there exists $c_i \in k^\times$ for $i = 1, 2, 3$ such that $A_i = c_i I_8$. Thus,

$$g_1 = \begin{pmatrix} A_0 & 0 & 0 & 0 \\ 0 & c_1 I_8 & 0 & 0 \\ 0 & 0 & c_2 I_8 & 0 \\ 0 & 0 & 0 & c_3 I_8 \end{pmatrix}.$$

Let $A_0 = (a_{ij})_{i,j=1,2,3}$, where $a_{ij} \in k$. Since $g_1 \in G_1$, there exists $c \in k^\times$ such that $\det(g_1 X) = c \det(X)$ for any $X \in \mathcal{J}$. Let

$$X_1 = \begin{pmatrix} 1 & t_3 & t_2 \\ t_3 & 0 & t_1 \\ t_2 & t_1 & 0 \end{pmatrix},$$

where $t_1, t_2, t_3 \in k$. Then,

$$\det(X_1) = 2t_1t_2t_3 - t_1^2.$$

We have

$$g_1X_1 = \begin{pmatrix} a_{11} & c_3t_3 & c_2t_2 \\ c_3t_3 & a_{21} & c_1t_1 \\ c_2t_2 & c_1t_1 & a_{31} \end{pmatrix},$$

$$\det(g_1X_1) = a_{11}a_{21}a_{31} + 2c_1c_2c_3t_1t_2t_3 - \sum_{i=1}^3 a_{i1}c_i^2t_i^2.$$

Since $\det(g_1X_1) = c \det(X_1)$ for any $t_1, t_2, t_3 \in k$ and $\text{ch}(k) \neq 2$, we have

$$(3.18) \quad c_1c_2c_3 = c \neq 0, \quad a_{11}c_1^2 = c, \quad a_{21}c_2^2 = a_{31}c_3^2 = 0.$$

So $a_{21} = a_{31} = 0$ since $c_2, c_3 \neq 0$.

Let

$$X_2 = \begin{pmatrix} 0 & t_3 & t_2 \\ t_3 & 1 & t_1 \\ t_2 & t_1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & t_3 & t_2 \\ t_3 & 0 & t_1 \\ t_2 & t_1 & 1 \end{pmatrix}.$$

Similarly as above, we have

$$(3.19) \quad a_{ij} = 0 \ (i \neq j), \quad a_{11}c_1^2 = a_{22}c_2^2 = a_{33}c_3^2 = c.$$

Let $g_2 = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ where $p, q, r, s \in k$. Since g fixes w , we have

$$(3.20) \quad \begin{aligned} a_{11}p &= 1, \quad -a_{22}p + a_{22}q = -1, \quad -a_{33}q = 0, \\ a_{11}r &= 0, \quad -a_{22}r + a_{22}s = 1, \quad -a_{33}s = -1. \end{aligned}$$

By (3.20) and (3.19), we have

$$a_{11} = a_{22} = a_{33}, \quad p = s = a_{11}^{-1}, \quad q = r = 0$$

and

$$(3.21) \quad c_1^2 = c_2^2 = c_3^2 = a_{11}^{-1}c.$$

By (3.18) and (3.21), we have $c_i = \pm a_{11}$ for $i = 1, 2, 3$. Let $\varepsilon_i = c_i/a_{11} \in \{\pm 1\}$ for $i = 1, 2, 3$. Since $c_1^2 = a_{11}^{-1}c = a_{11}^{-1}c_1c_2c_3$, we have $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$. Thus,

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, 1, 1), (1, -1, -1), (-1, 1, -1) \text{ or } (-1, -1, 1).$$

Therefore,

$$g = \left(a_{11} \left(\begin{array}{c|ccc} I_3 & & & 0 \\ \hline & \varepsilon_1 I_8 & 0 & 0 \\ 0 & 0 & \varepsilon_2 I_8 & 0 \\ & 0 & 0 & \varepsilon_3 I_8 \end{array} \right), a_{11}^{-1} I_2 \right).$$

Obviously, $(I_8, I_8, I_8), (-I_8, -I_8, I_8), (-I_8, I_8, -I_8)$ and $(I_8, -I_8, -I_8) \in M(= \text{Spin}(Q))$. Hence, we have $g \in G_w^\circ$. Thus, $G_w^\circ = \text{Ker}(\varphi)$ and the sequence (3.17) is exact. So $G_w/G_w^\circ \cong \mathfrak{S}_3$. We denote the restriction of φ on $\langle \tau_1, \tau_2 \rangle$ by $\varphi|_{\langle \tau_1, \tau_2 \rangle}$. Since $\varphi|_{\langle \tau_1, \tau_2 \rangle}$ is an isomorphism, $(\varphi|_{\langle \tau_1, \tau_2 \rangle})^{-1}$ is a section of φ . So the exact sequence (3.17) is split.

Therefore, the restriction of the natural homomorphism $\pi : G_w \rightarrow G_w/G_w^\circ$ on $\langle \tau_1, \tau_2 \rangle$ is an isomorphism defined over k . This completes the proof of Theorem 3.1. \square

Theorem 3.1 implies that the assumption on the stabilizer in Theorem 1.15 is satisfied.

Finally, we shall show that the diagram (3.16) is commutative. Since

$$q_{w,i}\tau_1 = \begin{cases} (0,1) \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} = (1,1) & i = 1 \\ (1,1) \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} = (0,1) & i = 2, \\ (1,0) \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} = (-1,0) & i = 3 \end{cases}$$

$$q_{w,i}\tau_2 = \begin{cases} (0,1) \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = (0,-1) & i = 1 \\ (1,1) \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = (1,0) & i = 2, \\ (1,0) \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = (1,1) & i = 3 \end{cases}$$

by (3.12), we have

$$(3.22) \quad \eta_w(\tau_1) = (1\ 2) = \varphi(\tau_1), \quad \eta_w(\tau_2) = (2\ 3) = \varphi(\tau_2).$$

It follows that $\eta_w(\tau) = \varphi(\tau)$ for all $\tau \in \langle \tau_1, \tau_2 \rangle$. For any $g \in G_w$, there exists $\tau \in \langle \tau_1, \tau_2 \rangle$ such that $\varphi(g) = \varphi(\tau)$. This implies that $g\tau^{-1} \in G_w^\circ$. So the $\mathrm{GL}(2)$ part of $g\tau^{-1}$ is a scalar matrix. Thus, $\eta_w(g\tau^{-1})$ is the unit element of \mathfrak{S}_3 . So we have $\eta_w(g) = \eta_w(\tau) = \varphi(\tau) = \varphi(g)$. Therefore, the diagram (3.16) is commutative.

4. RATIONAL ORBITS

In this section, we determine the arithmetic interpretation of the set of rational orbits. We have determined the stabilizer in the previous section. This and Theorem 1.15 enable us to reduce the consideration to that of Galois cohomology. Since $H^1(k, \mathrm{GL}(1)) = \{1\}$, we have to interpret the set $H^1(k, \mathrm{Spin}(Q) \rtimes \mathfrak{S}_3)$. We first present an alternative description of $\mathrm{Spin}(Q) \rtimes \mathfrak{S}_3$.

We denote the cubic subalgebra $\sum_{i=1}^3 kE_i$ of \mathcal{J} by \mathfrak{t} . We define an algebraic group $\mathrm{Aut}(\mathcal{J}, \mathfrak{t})$ by

$$\mathrm{Aut}(\mathcal{J}, \mathfrak{t}) = \{L \in F_4 \mid L(\mathfrak{t}) = \mathfrak{t}\}$$

(see (2.7) for the definition of F_4) and denote the Lie algebra of $\mathrm{Aut}(\mathcal{J}, \mathfrak{t})$ by $\mathfrak{a}(\mathcal{J}, \mathfrak{t})$. Then

$$\mathfrak{a}(\mathcal{J}, \mathfrak{t}) = \{L \in \mathfrak{f} \mid L(\mathfrak{t}) \subset \mathfrak{t}\}.$$

Lemma 4.1. $\mathfrak{a}(\mathcal{J}, \mathfrak{t}) = \mathfrak{D}_0$.

Proof. Let $X = X_0 + (\alpha)'_1 + (\beta)'_2 + (\gamma)'_3 \in \mathfrak{f} = \text{Der}_k(\mathcal{J}, \mathcal{J})$ where $X_0 \in \mathfrak{D}_0, \alpha, \beta, \gamma \in \mathcal{O}$. Since

$$X(E_i) = \begin{cases} \frac{1}{4}((\beta)_2 - (\gamma)_3) & i = 1 \\ \frac{1}{4}((\gamma)_3 - (\alpha)_1) & i = 2, \\ \frac{1}{4}((\alpha)_1 - (\beta)_2) & i = 3 \end{cases}$$

if $X \in \mathfrak{a}(\mathcal{J}, \mathfrak{t})$, then we have $\alpha = \beta = \gamma = 0$. Hence, we have $\mathfrak{a}(\mathcal{J}, \mathfrak{t}) \subset \mathfrak{D}_0$. The opposite inclusion is trivial and so $\mathfrak{a}(\mathcal{J}, \mathfrak{t}) = \mathfrak{D}_0$. \square

Let pr_1 be the natural projection from G to G_1 as in Introduction. Let $P = \text{pr}_1(\mathfrak{S}_3)$. Then it is easy to see that pr_1 induces an isomorphism $\mathfrak{S}_3 \cong P$. Since all elements in $\text{Spin}(Q)$ preserve three octonions \mathcal{O}_i where $i = 1, 2, 3$ (see (2.6)) in \mathcal{J} and such an element in P is only 1, $\text{Spin}(Q) \cap P = \{1\}$. So pr_1 induces an isomorphism $\text{Spin}(Q) \rtimes \mathfrak{S}_3 \cong \text{Spin}(Q) \rtimes P$. The following lemma is an alternative description of $\text{Spin}(Q) \rtimes \mathfrak{S}_3$.

Lemma 4.2. $\text{Spin}(Q) \rtimes P = \text{Aut}(\mathcal{J}, \mathfrak{t})$.

Proof. Since $\text{Spin}(Q)$ acts trivially on \mathfrak{t} (elements of $\text{Spin}(Q)$ are in the form (3.9) where $t = 1$) and all elements in P preserve \mathfrak{t} (see (3.10)), we have $(\text{Spin}(Q) \rtimes P) \subset \text{Aut}(\mathcal{J}, \mathfrak{t})$. It follows that

$$28 = \dim \text{Spin}(Q) \rtimes P \leq \dim \text{Aut}(\mathcal{J}, \mathfrak{t}) \leq \dim \mathfrak{a}(\mathcal{J}, \mathfrak{t}) = 28.$$

In particular, $\dim \text{Aut}(\mathcal{J}, \mathfrak{t}) = \dim \mathfrak{a}(\mathcal{J}, \mathfrak{t}) = 28$ and $\text{Aut}(\mathcal{J}, \mathfrak{t})$ is smooth. We have shown in the step (I) of the proof of Proposition 2.11 that $\mathfrak{D}_0 = \mathfrak{so}(Q)$. So by Lemma 4.1, $\text{Lie}(\text{Spin}(Q) \rtimes P) = \mathfrak{a}(\mathcal{J}, \mathfrak{t})$. Since both $\text{Spin}(Q) \subset \text{Aut}(\mathcal{J}, \mathfrak{t})$ are smooth, the identity component of $\text{Aut}(\mathcal{J}, \mathfrak{t})$ coincides with $\text{Spin}(Q)$. Therefore, we may assume that $k = \bar{k}$ and prove that $\text{Aut}(\mathcal{J}, \mathfrak{t})_{\bar{k}} \subset (\text{Spin}(Q) \rtimes P)_{\bar{k}}$ set-theoretically.

By the above consideration, $\text{Spin}(Q)$ is a normal subgroup of $\text{Aut}(\mathcal{J}, \mathfrak{t})$. Hence, we can define a homomorphism from $\text{Aut}(\mathcal{J}, \mathfrak{t})$ to $\text{Aut}(\text{Spin}(Q))$ by

$$\begin{array}{ccc} I : \text{Aut}(\mathcal{J}, \mathfrak{t}) & \rightarrow & \text{Aut}(\text{Spin}(Q)) \\ \downarrow & & \downarrow \\ g & \mapsto & I_g : X \mapsto gXg^{-1} \end{array}$$

for $X \in \text{Spin}(Q)$.

Since $\text{Spin}(Q) \rtimes P \subset \text{Aut}(\mathcal{J}, \mathfrak{t})$ and the restriction of I to $\text{Spin}(Q) \rtimes P$ is surjective, I is also surjective. Hence, for $g \in \text{Aut}(\mathcal{J}, \mathfrak{t})$, by multiplying an element of $\text{Spin}(Q) \rtimes P$, we may assume that I_g is trivial. Then, by the same argument as in Section 3, we have

$$g = \left(\begin{array}{ccc|ccc} a_1 & 0 & 0 & & & \\ 0 & a_2 & 0 & & & \\ 0 & 0 & a_3 & & & \\ \hline & & & 0 & & \\ & & & c_1 I_8 & 0 & 0 \\ & & & 0 & c_2 I_8 & 0 \\ & & & 0 & 0 & c_3 I_8 \end{array} \right),$$

where $a_i, c_i \in k^\times$ ($i = 1, 2, 3$) and

$$(4.3) \quad a_1 c_1^2 = a_2 c_2^2 = a_3 c_3^2 = c_1 c_2 c_3$$

(see (3.18), (3.19)).

Since $g \in \text{Aut}(\mathcal{J}, \mathfrak{t}) \subset F_4 = \text{Aut}(\mathcal{J})$, we have $g(e) = e$. So $a_1 = a_2 = a_3 = 1$. By (4.3), we have $c_1^2 = c_2^2 = c_3^2 = c_1 c_2 c_3$. It follows that

$$(c_1, c_2, c_3) = (1, 1, 1), (1, -1, -1), (-1, 1, -1) \text{ or } (-1, -1, 1).$$

Thus, $g \in \text{Spin}(Q)$. Therefore, $\text{Aut}(\mathcal{J}, \mathfrak{t})_{\bar{k}} \subset (\text{Spin}(Q) \rtimes P)_{\bar{k}}$. \square

We shall consider the interpretation of $H^1(k, G_w)$. We have shown in Section 3 that $G_w \cong \text{GL}(1) \times (\text{Spin}(Q) \rtimes \mathfrak{S}_3)$. Note that $H^1(k, \text{GL}(n)) = \{1\}$ for all n . So $H^1(k, G_w), H^1(k, G)$ can be identified with $H^1(k, \text{Spin}(Q) \rtimes \mathfrak{S}_3), H^1(k, G_1)$ respectively. So by Lemma 4.2, we may consider $\text{Ker}(H^1(k, \text{Aut}(\mathcal{J}, \mathfrak{t})) \rightarrow H^1(k, G_1))$.

Let \mathcal{M} be a Jordan algebra over k and \mathfrak{n} be a cubic étale subalgebra of \mathcal{M} over k . If there exists an isomorphism between \mathcal{M} and \mathcal{J} over k^{sep} which induces an isomorphism between \mathfrak{n} and \mathfrak{t} over k^{sep} , we call $(\mathcal{M}, \mathfrak{n})$ a k -form of $(\mathcal{J}, \mathfrak{t})$. Two k -forms $(\mathcal{J}_1, \mathfrak{t}_1), (\mathcal{J}_2, \mathfrak{t}_2)$ are said to be equivalent if there exists an isomorphism between \mathcal{J}_1 and \mathcal{J}_2 over k which induces an isomorphism between \mathfrak{t}_1 and \mathfrak{t}_2 over k . Note that even if two cubic étale subalgebras \mathfrak{t}_1 and \mathfrak{t}_2 are isomorphic over k , two k -forms $(\mathcal{J}, \mathfrak{t}_1)$ and $(\mathcal{J}, \mathfrak{t}_2)$ may not be equivalent.

We shall give a correspondence between $H^1(k, \text{Aut}(\mathcal{J}, \mathfrak{t}))$ and the set of k -forms of $(\mathcal{J}, \mathfrak{t})$.

Suppose that \mathcal{M} is a k -form of \mathcal{J} . For $\sigma \in \text{Gal}(k^{\text{sep}}/k)$, let $v_{\mathcal{M}}(\sigma)$ be the semi-linear homomorphism

$$v_{\mathcal{M}}(\sigma) : \mathcal{M}_{k^{\text{sep}}} = k^{\text{sep}} \otimes \mathcal{M} \ni \sum a_i \otimes m_i \mapsto \sum a_i^\sigma \otimes m_i \in k^{\text{sep}} \otimes \mathcal{M}.$$

We define a map

$$(4.4) \quad A : H^1(k, \text{Aut}(\mathcal{J}, \mathfrak{t})) \rightarrow \{\text{equivalence classes of } k\text{-forms of } (\mathcal{J}, \mathfrak{t})\}$$

as follows.

Let $c \in H^1(k, \text{Aut}(\mathcal{J}, \mathfrak{t}))$ be the element determined by a 1-cocycle h with coefficients in $\text{Aut}(\mathcal{J}, \mathfrak{t})_{k^{\text{sep}}}$. We define

$$\mathcal{M}(h) = \{X \in \mathcal{J}_{k^{\text{sep}}} \mid h_1(\sigma) v_{\mathcal{J}}(\sigma)(X) = X\},$$

$$\mathfrak{n}(h) = \{Y \in \mathfrak{t}_{k^{\text{sep}}} \mid h_1(\sigma) v_{\mathcal{J}}(\sigma)(Y) = Y\}.$$

Then, $\mathcal{M}(h), \mathfrak{n}(h)$ are closed under the addition and the multiplication of $\mathcal{J}_{k^{\text{sep}}}$ since $h(\sigma) \in F_{4, k^{\text{sep}}}$ and the action of the Galois group commutes with the addition and the multiplication of $\mathcal{J}_{k^{\text{sep}}}$. So $(\mathcal{M}(h), \mathfrak{n}(h))$ is a k -form of $(\mathcal{J}, \mathfrak{t})$. We define $A(h) = (\mathcal{M}(h), \mathfrak{n}(h))$. One can show by the standard argument that $A(h)$ depends only on the cohomology class c . By abuse of notation, we use the notation $A(c)$ also.

Next, we define a map

$$B : \{\text{equivalence classes of } k\text{-forms of } (\mathcal{J}, \mathfrak{t})\} \rightarrow H^1(k, \text{Aut}(\mathcal{J}, \mathfrak{t}))$$

as follows.

Let $(\mathcal{M}, \mathfrak{n})$ be a k -form of $(\mathcal{J}, \mathfrak{t})$. Let $\phi : \mathcal{M}_{k^{\text{sep}}} \rightarrow \mathcal{J}_{k^{\text{sep}}}$ be an isomorphism of Jordan algebras which induces an isomorphism from $\mathfrak{n}_{k^{\text{sep}}}$ to $\mathfrak{t}_{k^{\text{sep}}}$. Then we define a map $h_{\mathcal{M}}$ from $\text{Gal}(k^{\text{sep}}/k)$ to $\text{GL}(\mathcal{J})_{k^{\text{sep}}}$ by

$$h_{\mathcal{M}}(\sigma) = \phi v_{\mathcal{M}}(\sigma) \phi^{-1} v_{\mathcal{J}}(\sigma)^{-1}.$$

Then one can show by an easy computation, that $h_{\mathcal{M}}$ is 1-cocycle with coefficients in $\text{Aut}(\mathcal{J}, \mathfrak{t})_{k^{\text{sep}}}$. Let $B((\mathcal{M}, \mathfrak{n}))$ be the element of $H^1(k, \text{Aut}(\mathcal{J}, \mathfrak{t}))$ determined by $h_{\mathcal{M}}$. This definition is also well-defined.

The proof of the following proposition is standard and is left to the reader.

Proposition 4.5. *The maps A, B are the inverses of each other and so*

$$A : H^1(k, G_w) \rightarrow \{\text{equivalence classes of } k\text{-forms of } (\mathcal{J}, \mathfrak{t})\}$$

is bijective.

We next consider the interpretation of the set $\text{Ker}(H^1(k, G_w) \rightarrow H^1(k, G_1))$.

We first define the notion of *isotopes* of \mathcal{J} . For $m \in \mathcal{J}$ such that $\det(m) \neq 0$, we can associate a Jordan algebra \mathcal{J}_m related to m as follows. The underlying vector space of \mathcal{J}_m is \mathcal{J} . We define

$$\begin{aligned} \langle x, y \rangle_m &= -6 \det(m)^{-1} D(x, y, m) + 9 \det(m)^{-2} D(x, m, m) D(y, m, m), \\ x \circ_m y &= 4 \det(m)^{-1} (x \times m) \times (y \times m) + \frac{1}{2} (\langle x, y \rangle_m - \langle x, m \rangle_m \langle y, m \rangle_m) m \end{aligned}$$

for $x, y \in \mathcal{J}$. It is known that \mathcal{J}_m is a Jordan algebra with $x \circ_m y$, $\langle x, x \rangle_m$ and m as the product, the norm and the unit element. Moreover, there is an isomorphism of Jordan algebras $\mathcal{J}_m \otimes k^{\text{sep}} \rightarrow \mathcal{J} \otimes k^{\text{sep}}$. This construction is given in [12, p.155].

Note that if the octonion is split, then all isotopes of \mathcal{J} are isomorphic to \mathcal{J} over k (it follows from the three facts [p. 155, Proposition 5.9.2], [p. 147, Proposition 5.6.2] and [p. 153, Proposition 5.8.2]).

For an arbitrary k -form \mathcal{M} of \mathcal{J} , the notion of the determinant on \mathcal{M} is defined as follows. Let $\phi : \mathcal{M}_{k^{\text{sep}}} \rightarrow \mathcal{J}_{k^{\text{sep}}}$ be an isomorphism. For any $X \in \mathcal{M}$, we define $\det_{\mathcal{M}}$ on \mathcal{M} by $\det_{\mathcal{M}}(X) = \det(\phi(X))$. Since automorphisms of \mathcal{J} preserve the determinant on \mathcal{J} (see [12, p.133, Proposition 5.3.10]), $\det_{\mathcal{M}}$ is independent of the choice of the isomorphism ϕ and is defined over k .

It is also known that the determinant \det_m on \mathcal{J}_m satisfies the following equation:

$$(4.6) \quad \det_m(x) = \det(m)^{-1} \det(x)$$

for $x \in \mathcal{J}$ (see [12, p.155, Proposition 5.9.2]).

Let $\text{JIC}(k)$ be the set of equivalence classes of pairs $(\mathcal{J}_m, \mathfrak{n})$ of isotopes of \mathcal{J} and their cubic étale subalgebras. We show that elements in $\text{JIC}(k)$ correspond to elements in $\text{Ker}(H^1(k, G_w) \rightarrow H^1(k, G))$ bijectively.

Let h be a 1-cocycle with coefficients in $G_w k^{\text{sep}}$ which corresponds to $(\mathcal{M}, \mathfrak{n})$. We write $h = (h_1, h_2)$ where $h(\sigma) = (h_1(\sigma), h_2(\sigma))$ and $h_1(\sigma) \in G_1 k^{\text{sep}}, h_2(\sigma) \in \text{GL}(2)_{k^{\text{sep}}}$.

Let $c \in H^1(k, G_w)$ be the cohomology class determined by h and $A(c) = (\mathcal{M}, \mathfrak{n})$. Suppose that $c \in \text{Ker}(H^1(k, G_w) \rightarrow H^1(k, G))$. Since h is trivial in $H^1(k, G)$, there

exists $g = (g_1, g_2) \in G_{k^{\text{sep}}}$ such that

$$h(\sigma) = g^{-1}g^\sigma$$

for any $\sigma \in \text{Gal}(k^{\text{sep}}/k)$.

For $g = (g_1, g_2) \in G$, we define an element of G_1 by

$$(4.7) \quad \mu_g = c(g_1) \det(g_2)^2 g_1 \in G_1.$$

Note that the map $g \rightarrow \mu_g$ is a homomorphism. If $g \in G_{w k^{\text{sep}}}$, then by Theorem 3.1 and its proof, there exist $t \in k, u \in \text{Spin}(Q), \tau \in \langle \tau_1, \tau_2 \rangle$ such that $g = (tu, t^{-1}I_2)\tau$. So by computation,

$$(4.8) \quad \mu_g = c(tu \text{pr}_1(\tau))(t^{-4})tu \text{pr}_1(\tau) = u \text{pr}_1(\tau)$$

where pr_1 is the natural projection from G to G_1 . It follows that $\mu_g \in (\text{Spin}(Q) \rtimes P)_{k^{\text{sep}}} = \text{Aut}(\mathcal{J}, \mathfrak{t})_{k^{\text{sep}}}$.

For the above 1-cocycle h , since $a = (c(g_1) \det(g_2)^2 I_2, (c(g_1) \det(g_2)^2)^{-1} I_2) \in G_{w k^{\text{sep}}}$, we may substitute $a^{-1}h(\sigma)a^\sigma$ for $h(\sigma)$ for any $\sigma \in \text{Gal}(k^{\text{sep}}/k)$. Then $h_1(\sigma) = \mu_g^{-1}\mu_g^\sigma = \mu_{g^{-1}g^\sigma}$.

We show that there exists $m \in \mathcal{J}$ with $\det(m) \neq 0$ such that \mathcal{M} is isomorphic to \mathcal{J}_m . Let $\mu_g(e) = m$. Since $g^{-1}g^\sigma \in G_{w k^{\text{sep}}}$, $\mu_{g^{-1}g^\sigma} \in \text{Aut}(\mathcal{J}, \mathfrak{t})_{k^{\text{sep}}} \subset F_4 k^{\text{sep}}$. Therefore, we have $\mu_g^{-1}\mu_g^\sigma(e) = e$. So

$$\begin{aligned} v_{\mathcal{J}}(\sigma)(m) &= v_{\mathcal{J}}(\sigma)(\mu_g(e)) = v_{\mathcal{J}}(\sigma)(\mu_g(v_{\mathcal{J}}(\sigma)^{-1}(v_{\mathcal{J}}(\sigma)(e)))) \\ &= \mu_g^\sigma(v_{\mathcal{J}}(\sigma)(e)) = \mu_g^\sigma(e) = \mu_g(e) = m \end{aligned}$$

for any $\sigma \in \text{Gal}(k^{\text{sep}}/k)$. Hence, we have $m \in \mathcal{J}$.

It is known that all isotopes are isomorphic over k^{sep} (it follows from [12, p.158, Proposition 5.9.3] and the proof of [12, p.181, Theorem 7.3.2]). So there exists an isomorphism $\phi : \mathcal{J}_{k^{\text{sep}}} \rightarrow \mathcal{J}_m k^{\text{sep}}$. Since $\phi^{-1}\mu_g(e) = e$, by (2.8), $\phi^{-1}\mu_g \in F_4 k^{\text{sep}}$. Thus, $\mu_g (= \phi\phi^{-1}\mu_g)$ is an isomorphism from $\mathcal{J}_{k^{\text{sep}}}$ to $\mathcal{J}_m k^{\text{sep}}$. Since

$$\begin{aligned} (4.9) \quad \mathcal{M} &= \{X \in \mathcal{J}_{k^{\text{sep}}} \mid \mu_g^\sigma(v_{\mathcal{J}}(\sigma)(X)) = \mu_g(X), \quad \forall \sigma \in \text{Gal}(k^{\text{sep}}/k)\} \\ &= \{X \in \mathcal{J}_{k^{\text{sep}}} \mid v_{\mathcal{J}}(\sigma)(\mu_g(X)) = \mu_g(X), \quad \forall \sigma \in \text{Gal}(k^{\text{sep}}/k)\} , \\ &= \mu_g^{-1}(\mathcal{J}) \end{aligned}$$

the k -linear map $\mu_g : \mathcal{M} \rightarrow \mathcal{J}_m$ is a k -linear isomorphism. Therefore, $\mathcal{M} \cong \mathcal{J}_m$ as Jordan algebras over k .

Conversely, suppose that $m \in \mathcal{J}$, $\det(m) \neq 0$ and \mathfrak{n} is a cubic étale subalgebra of \mathcal{J}_m . We show that $B((\mathcal{J}_m, \mathfrak{n})) \in \text{Ker}(H^1(k, G_w) \rightarrow H^1(k, G))$.

If $k = k^{\text{sep}}$, then all cubic étale algebras are isomorphic to $(k^{\text{sep}})^3$. So there exist three idempotents u_1, u_2, u_3 such that u_i 's are pairwise orthogonal and $u_1 + u_2 + u_3$ is the unit element. So for any cubic étale subalgebras $\mathfrak{n}_1, \mathfrak{n}_2$ of \mathcal{J} , there exist $u_{i1}, u_{i2}, u_{i3} \in \mathfrak{n}_i k^{\text{sep}}$ for $i = 1, 2$ which satisfy the above conditions. Since \mathcal{O} is split if $k = k^{\text{sep}}$, the map $\mathcal{O}_{k^{\text{sep}}}^\times \ni x \mapsto \|x\| \in (k^{\text{sep}})^\times$ is surjective. In the situation where $k = k^{\text{sep}}$, by [1, p.413, THEOREM 9], there exists $\phi \in F_4 k^{\text{sep}}$ such that $\phi(u_{1j}) = u_{2j}$ for $j = 1, 2, 3$. Then $\phi(\mathfrak{n}_1 k^{\text{sep}}) = \mathfrak{n}_2 k^{\text{sep}}$.

For any $(\mathcal{J}_m, \mathfrak{n})$, there exists an isomorphism $\phi_1 : \mathcal{J}_m k^{\text{sep}} \rightarrow \mathcal{J}_{k^{\text{sep}}}$. By the above argument, there exists $\phi_2 \in F_4 k^{\text{sep}}$ such that $\phi_2(\phi_1(\mathfrak{n}_{k^{\text{sep}}})) = \mathfrak{t}_{k^{\text{sep}}}$. Then $\phi_2\phi_1 :$

$\mathcal{J}_m k^{\text{sep}} \rightarrow \mathcal{J}_k^{\text{sep}}$ is an isomorphism which induces an isomorphism from $\mathfrak{n}_k^{\text{sep}}$ to $\mathfrak{t}_k^{\text{sep}}$. So $(\mathcal{J}_m, \mathfrak{n})$ is a k -form of $(\mathcal{J}, \mathfrak{t})$. Moreover, the map $\sigma \mapsto \phi v_{\mathcal{J}_m}(\sigma) \phi^{-1} v_{\mathcal{J}}(\sigma)^{-1}$ is a 1-cocycle which defines the cohomology class $B((\mathcal{J}_m, \mathfrak{n}))$.

Since the determinant on $\mathcal{J}_m k^{\text{sep}}$ satisfies (4.6) and $\det(\phi(X)) = \det_m(X)$ for any $X \in \mathcal{J}_m k^{\text{sep}}$, we have $\phi \in G_1 k^{\text{sep}}$. It follows that $B((\mathcal{J}_m, \mathfrak{n})) \in \text{Ker} (H^1(k, G_w) \rightarrow H^1(k, G))$. Thus, elements in $\text{JIC}(k)$ correspond to elements in $\text{Ker} (H^1(k, G_w) \rightarrow H^1(k, G_1))$ bijectively.

Therefore, by the above argument and Theorem 1.15, we have the following interpretation of the set of rational orbits.

Theorem 4.10. *The following map*

$$G_k \backslash V_k^{\text{ss}} \ni x \mapsto A(c_x) \in \text{JIC}(k)$$

(see (1.15) for the definition of c_x) is bijective.

5. EQUIVARIANT MAP

In Section 4, we have shown that elements of the set $\text{Ker} (H^1(k, G_w) \rightarrow H^1(k, G))$ are in bijective correspondence with equivalence classes of pairs $(\mathcal{J}_m, \mathfrak{n})$ of isotopes of \mathcal{J} and their cubic subalgebras. In this section, for any $x \in V_k^{\text{ss}}$, we construct explicitly the isotope of \mathcal{J} and its cubic subalgebra corresponding to x in Theorem 4.10 by an equivariant map from V to \mathcal{J} .

Before defining an equivariant map, we review some properties of the cross product. The following formulas are proved in [12, p.154] and [12, pp.122-123, Lemma 5.1.2].

Lemma 5.1. *The following equations are satisfied for all $x, y \in \mathcal{J}$.*

- (i) $x \times y = x \circ y - \frac{1}{2} \langle y, e \rangle x - \frac{1}{2} \langle x, e \rangle y - \frac{1}{2} \langle x, y \rangle e + \frac{1}{2} \langle x, e \rangle \langle y, e \rangle e$.
- (ii) $x \circ (x \times x) = \det(x) e$ (see (2.3) also).
- (iii) $(x \times x) \times (x \times x) = \det(x) x$.
- (iv) $4x \times (y \times (x \times x)) = \det(x) y + \langle x, y \rangle x \times x$.
- (v) $4(x \times x) \times (x \times y) = \det(x) y + 3D(x, x, y) x$.
- (vi) $4(x \times y) \times (x \times y) = -2(x \times x) \times (y \times y) + 3D(x, y, y) x + 3D(x, x, y) y$.

For any $g \in \text{GL}(\mathcal{J})$, we define $\tilde{g} \in \text{GL}(\mathcal{J})$ by

$$\langle g(x), \tilde{g}(y) \rangle = \langle x, y \rangle, \quad \forall x, y \in \mathcal{J}.$$

The following lemma is proved in [12, p.180, Proposition 7.3.1].

Lemma 5.2. *The map $g \mapsto \tilde{g}$ is an automorphism of H_1 with order 2 and the following equations:*

$$g(x \times y) = \tilde{g}(x) \times \tilde{g}(y), \quad \tilde{g}(x \times y) = g(x) \times g(y) \quad \forall x, y \in \mathcal{J}$$

are satisfied.

We define a map $m : V = \mathcal{J} \otimes \text{Aff}^2 \rightarrow \mathcal{J}$ by

$$m(x_1 v_1 + x_2 v_2) = 6(x_1 \times x_1) \times (x_2 \times x_2) - 3D(x_1, x_2, x_2) x_1 - 3D(x_1, x_1, x_2) x_2.$$

We define an action of G on \mathcal{J} by

$$(5.3) \quad G \times \mathcal{J} \ni ((g_1, g_2), x) \mapsto c(g_1)(\det(g_2))^2 g_1(x) \in \mathcal{J}.$$

Proposition 5.4. *For any $(g_1, g_2) \in G$ and $x_1 v_1 + x_2 v_2 \in V$,*

$$m((g_1, g_2)(x_1 v_1 + x_2 v_2)) = c(g_1) \det(g_2)^2 g_1(m(x_1 v_1 + x_2 v_2)).$$

Proof. We first prove that the map m is $\mathrm{GL}(2)$ -equivariant. Since $\mathrm{GL}(2)$ is generated by upper triangular matrices and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, it is enough to consider these two cases. We denote $m(x_1 v_1 + x_2 v_2)$ by $m(x_1, x_2)$. For any upper triangular matrix $g_2 = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$,

$$\begin{aligned} m(ax_1 + bx_2, dx_2) &= 6((ax_1 + bx_2) \times (ax_1 + bx_2)) \times (dx_2 \times dx_2) \\ &\quad - 3D(ax_1 + bx_2, dx_2, dx_2)(ax_1 + bx_2) \\ &\quad - 3D(ax_1 + bx_2, ax_1 + bx_2, dx_2)dx_2 \\ &= a^2 d^2 \{6(x_1 \times x_1) \times (x_2 \times x_2) \\ &\quad - 3D(x_1, x_2, x_2)x_1 - 3D(x_1, x_1, x_2)x_2\} \\ &\quad + 12abd^2(x_1 \times x_2) \times (x_2 \times x_2) + 6b^2 d^2(x_2 \times x_2) \times (x_2 \times x_2) \\ &\quad - 3abd^2 D(x_2, x_2, x_2)x_1 - 9abd^2 D(x_1, x_2, x_2)x_2 \\ &\quad - 6b^2 d^2 D(x_2, x_2, x_2)x_2 \\ &= a^2 d^2 m(x_1, x_2) \\ &\quad + 3abd^2 \{4(x_1 \times x_2) \times (x_2 \times x_2) - \det(x_2)x_1 - 3D(x_1, x_2, x_2)x_2\} \\ &= a^2 d^2 m(x_1, x_2) \text{ (by (v))} \\ &= (\det(g_2))^2 m(x_1, x_2) \end{aligned}$$

For $g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, by definition, $m(g_2(x_1 v_1 + x_2 v_2)) = m(x_2, x_1) = m(x_1, x_2)$. Therefore, m is $\mathrm{GL}(2)$ -equivariant.

We next prove that m is G_1 -equivariant. For any $g \in H_1$, by Lemma 5.2,

$$\begin{aligned} g((x \times y) \times (z \times w)) &= (\tilde{g}(x \times y)) \times (\tilde{g}(z \times w)) \\ &= (g(x) \times g(y)) \times (g(z) \times g(w)) \end{aligned}$$

for $x, y, z, w \in \mathcal{J}$. For any $g_1 \in G_{1k^{\mathrm{sep}}}$, since $\mathrm{ch}(k) \neq 3$, there exists $t \in k^{\mathrm{sep}}$ such that $t^3 = c(g_1)$. Then $t^{-1}g_1 \in H_{1k^{\mathrm{sep}}}$. So

$$\begin{aligned} g_1((x \times y) \times (z \times w)) &= t(t^{-1}g_1)((x \times y) \times (z \times w)) \\ &= t \left(t^{-1}g_1(x) \times t^{-1}g_1(y) \right) \times \left(t^{-1}g_1(z) \times t^{-1}g_1(w) \right) \\ &= c(g_1)^{-1} (g_1(x) \times g_1(y)) \times (g_1(z) \times g_1(w)) \end{aligned}$$

for $x, y, z, w \in \mathcal{J}$. It follows that

$$\begin{aligned} g_1(m(x_1, x_2)) &= 6g_1((x_1 \times x_1) \times (x_2 \times x_2)) \\ &\quad - 3D(x_1, x_2, x_2)g_1(x_1) - 3D(x_1, x_1, x_2)g_1(x_2) \\ &= c(g)^{-1} (m(g_1(x_1), g_1(x_2))). \end{aligned}$$

Therefore, for any $(g_1, g_2) \in G$, we have

$$(5.5) \quad m((g_1, g_2)(x_1 v_1 + x_2 v_2)) = c(g_1) \det(g_2)^2 g_1(m(x_1 v_1 + x_2 v_2)).$$

□

For $x \in V_k^{\text{ss}}$, to define the isotope $\mathcal{J}_{m(x)}$ of \mathcal{J} , $\det(m(x))$ must not be equal to 0. So we need the following lemma.

Lemma 5.6. *For $w = w_1v_1 + w_2v_2 \in V$ in (1.13), $m(w) = e$. In particular, $\mathcal{J}_{m(w)} = \mathcal{J}$.*

Proof. By direct computation, we have

$$\begin{aligned} w_1 \times w_1 &= -E_3, \quad w_2 \times w_2 = -E_1, \quad (w_1 \times w_1) \times (w_2 \times w_2) = \frac{1}{2}E_2, \\ D(w_1, w_1, w_2) &= \frac{1}{3}, \quad D(w_1, w_2, w_2) = -\frac{1}{3}, \end{aligned}$$

(see (2.4) for the definition of E_1, E_2, E_3). So

$$\begin{aligned} m(w) &= 6(w_1 \times w_1) \times (w_2 \times w_2) - 3D(w_1, w_2, w_2)w_1 - 3D(w_1, w_1, w_2)w_2 \\ &= 3E_2 + w_1 - w_2 \\ &= e. \end{aligned}$$

It is known that the multiplication in \mathcal{J}_e is the same as that of \mathcal{J} (see [12, p.158]). So $\mathcal{J}_{m(w)} = \mathcal{J}$. □

By the above lemma and Proposition 5.4, the map $x \mapsto \det(m(x))$ is a non-trivial relative invariant polynomial of degree 12. So $x \in V^{\text{ss}}$ if and only if $\det(m(x)) \neq 0$. Thus, for all $x \in V_k^{\text{ss}}$, we can define the isotope $\mathcal{J}_{m(x)}$ of \mathcal{J} .

We next define a subspace $\mathfrak{t}(x)$ of \mathcal{J} which is a cubic étale k -subalgebra with respect to the multiplication in $\mathcal{J}_{m(x)}$ for $x = x_1v_1 + x_2v_2 \in V_k^{\text{ss}}$. We shall show that the pair $(\mathcal{J}_{m(x)}, \mathfrak{t}(x))$ is a k -form of $(\mathcal{J}, \mathfrak{t})$ which is equivalent to $A(c_x)$ in Theorem 4.10.

Let $\mathfrak{t}(x)$ be the vector space spanned by x_1, x_2 and $m(x)$. Since $m(w) = e$, it is easy to see that $\mathfrak{t}(w) = \mathfrak{t}$. For any $g_1 \in G_1$, by Proposition 5.4,

$$\mathfrak{t}((g_1, I_2)(x)) = g_1(\mathfrak{t}(x)).$$

Moreover, it is easy to see that $\mathfrak{t}(x)$ is $\text{GL}(2)$ -invariant. Hence, for any $g \in G$, we have

$$\mathfrak{t}(g(x)) = g_1(\mathfrak{t}(x)).$$

We show that $\mathfrak{t}(x)$ is a cubic étale subalgebra of $\mathcal{J}_{m(x)}$. For $g = (g_1, g_2) \in G_{k^{\text{sep}}}$, let $\mu_g = c(g_1) \det(g_2)^2 g_1 \in G_{1, k^{\text{sep}}}$ (see (4.7) also). For any $x = x_1v_1 + x_2v_2 \in V_k^{\text{ss}}$, there exists $g = (g_1, g_2) \in G_{k^{\text{sep}}}$ such that $gw = x$. Then,

$$(5.7) \quad m(x) = m(gw) = c(g_1) \det(g_2)^2 g_1(m(w)) = \mu_g(e).$$

By the same argument as the one just before (4.9), the equation (5.7) implies that the linear map $\mu_g : \mathcal{J}_{k^{\text{sep}}} \rightarrow \mathcal{J}_{m(x), k^{\text{sep}}}$ is an isomorphism.

Moreover, since

$$\mathfrak{t}(x)_{k^{\text{sep}}} = \mathfrak{t}(gw)_{k^{\text{sep}}} = g_1(\mathfrak{t}(w)_{k^{\text{sep}}}) = g_1(\mathfrak{t}_{k^{\text{sep}}}) = \mu_g(\mathfrak{t}_{k^{\text{sep}}}),$$

$\mathfrak{t}(x)_{k^{\text{sep}}}$ is a cubic étale subalgebra of $\mathcal{J}_{m(x), k^{\text{sep}}}$. Since the multiplication in $\mathcal{J}_{m(x), k^{\text{sep}}}$ is defined over k , $\mathfrak{t}(x)$ is a cubic étale subalgebra of $\mathcal{J}_{m(x)}$. Moreover, $(\mathcal{J}_{m(x)}, \mathfrak{t}(x))$ is a k -form of $(\mathcal{J}, \mathfrak{t})$.

By the argument above, the pair $(\mathcal{J}_{m(x)}, \mathfrak{t}(x))$ corresponds to the cohomology class $c_x \in H^1(k, G_w)$ (see (1.14) for the definition of c_x). Therefore, we have the following theorem.

Theorem 5.8. *The following map is bijective.*

$$G_k \backslash V_k^{\text{ss}} \ni x \mapsto (\mathcal{J}_{m(x)}, \mathfrak{t}(x)) \in \text{JIC}(k).$$

Since the product structure on \mathcal{J}_m is defined by homogeneous polynomials of degree 11, the product structure on $\mathcal{J}_{m(x)}$ is defined by homogeneous polynomials of degree 44. However, it is possible to construct an equivariant map from V to $\text{Hom}_k(\mathcal{J} \otimes \mathcal{J}, \mathcal{J})$ defined by homogeneous polynomials of degree 8. For this, see [7].

6. THE SPLIT CASE

The purpose of this section is to prove Theorem 1.19. For that purpose we need some preparation.

When we consider actions of $\text{Gal}(k^{\text{sep}}/k)$ on \mathfrak{S}_n (n is a positive integer) in this section, we always consider the trivial action of the Galois group. So if $\{h(\sigma)\}$ is a 1-cocycle with coefficients in \mathfrak{S}_n , then $\text{Gal}(k^{\text{sep}}/k) \ni \sigma \mapsto h(\sigma)^{-1} \in \mathfrak{S}_n$ is a homomorphism by Definition 2.2. We identify $H^1(k, \mathfrak{S}_n)$ with conjugacy classes of homomorphisms from $\text{Gal}(k^{\text{sep}}/k)$ to \mathfrak{S}_n in this manner.

We use the space of pairs of ternary quadratic forms in order to prove Theorem 1.19. The interpretation of rational orbits in this case was carried out in [14]. We first review this case and point out properties which will be needed in this section.

Let \mathcal{V}_1 be the space of ternary quadratic forms. Since $\text{ch}(k) \neq 2$, we may identify \mathcal{V}_1 space of symmetric 3×3 matrices. Let $\mathcal{G}_1 = \text{GL}(3)$. If $g \in \mathcal{G}_1$ and $x \in \mathcal{V}_1$, then we define $\rho_1(g)x = gx^t g$. This defines a representation of \mathcal{G}_1 on \mathcal{V}_1 . Let $\mathcal{G} = \text{GL}(3) \times \text{GL}(2)$. Regarding Aff^2 as the standard representation of $\text{GL}(2)$, $\mathcal{V} = \mathcal{V}_1 \otimes \text{Aff}^2$ is a representation of \mathcal{G} . We use the notation $\rho(g)$ for this representation. The representations ρ_1, ρ can be extended to representations on \mathcal{J}, V respectively by the same formulas. Note that elements of k commute with elements of \mathcal{O} and so the order of multiplication does not matter. We later show (Corollary 6.19) that $\rho_1(g_1) \in G_1$ (resp. $\rho(g) \in G$) for $g_1 \in \text{GL}(3)$ (resp. $g \in \mathcal{G}$).

We express elements of \mathcal{V} as $x = (x_1, x_2)$ ($x_1, x_2 \in \mathcal{V}_1$) or $x = x_1 v_1 + x_2 v_2$ using variables $v = (v_1, v_2)$. If $x \in \mathcal{V}_1$, the determinant defined by (1.4) coincides with the usual determinant. It is well-known that $(\mathcal{G}, \mathcal{V})$ is a regular prehomogeneous vector space and if we express elements of \mathcal{V} as $x = x_1 v_1 + x_2 v_2$, then the discriminant of the cubic form $F_x(v)$ (see (1.11)) is a relative invariant polynomial. Therefore, $x \in \mathcal{V}$ is semi-stable with respect to the action of \mathcal{G} if and only if it is semi-stable as an element of V with respect to the action of G .

If $x = (x_1, x_2) \in \mathcal{V}_k$, then x_1, x_2 define two conics in \mathbb{P}_k^2 . Let $\text{Zero}_{\mathbb{P}^2}(x) \subset \mathbb{P}_k^2$ be the intersection of these two conics. This should not be confused with $\text{Zero}(x) \subset \mathbb{P}_k^1$ (see (3.13)). If $g = (g_1, g_2) \in \mathcal{G}_{\bar{k}}$, $x, y \in \mathcal{V}_{\bar{k}}^{\text{ss}}$ and $x = gy$, then it is easy to see that $\text{Zero}_{\mathbb{P}^2}(x) = \text{Zero}_{\mathbb{P}^2}(y)g_1^{-1}$. In particular, if $g = (g_1, g_2) \in \mathcal{G}_{x\bar{k}}$, then $\text{Zero}_{\mathbb{P}^2}(x) = \text{Zero}_{\mathbb{P}^2}(x)g_1^{-1}$ and so $\text{Zero}_{\mathbb{P}^2}(x) = \text{Zero}_{\mathbb{P}^2}(x)g_1$ also.

The following proposition is proved in [14, pp.290, Propositions 1.5, 1.6].

Proposition 6.1. *Suppose that $x \in \mathcal{V}_k$. Then $x \in \mathcal{V}_k^{\text{ss}}$ if and only if $\text{Zero}_{\mathbb{P}^2}(x)$ consists of four distinct points. Moreover, if $x \in \mathcal{V}_k^{\text{ss}}$ then the coordinates of points in $\text{Zero}_{\mathbb{P}^2}(x)$ belong to k^{sep} .*

Note that the second statement is now a consequence of [15].

If $x \in \mathcal{V}_k^{\text{ss}}$, $\text{Zero}_{\mathbb{P}^2}(x) = \{p_{x,1}, \dots, p_{x,4}\}$ and $g \in \mathcal{G}_{x\bar{k}}$, then let $\gamma_x(g) \in \mathfrak{S}_4$ be the element such that

$$(6.2) \quad p_{x,j}g = p_{x,\gamma_x(g)(j)}$$

for $j = 1, \dots, 4$. Then $\gamma_x : \mathcal{G}_{x\bar{k}} \ni g \mapsto \gamma_x(g) \in \mathfrak{S}_4$ is a homomorphism.

The element w in Section 1 of [14] is not w in this paper, but is the following element:

$$w' = \frac{1}{2} \left(\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right).$$

Let

$$g = \left(\frac{1}{2} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \right) \in \mathcal{G}_k.$$

Then $gw = w'$.

It is easy to see that $\text{Zero}_{\mathbb{P}^2}(w) = \{p_{w,1}, \dots, p_{w,4}\}$ where

$$p_{w,1} = (1, -1, -1), p_{w,2} = (-1, 1, -1), p_{w,3} = (-1, -1, 1), p_{w,4} = (1, 1, 1).$$

We denote τ_1, τ_2, τ_3 in [14, p.289] by $\overline{\tau}_1, \overline{\tau}_2, \overline{\tau}_3$ in this paper. It is easy to see that $\overline{\tau}_1, \overline{\tau}_2, \overline{\tau}_3$ generate a subgroup of $\mathcal{G}_{w'k}$ which is isomorphic to \mathfrak{S}_4 and maps isomorphically to $\mathcal{G}_{w'k}/\mathcal{G}_{w'k}^\circ$. Moreover, $\gamma_{w'}$ induces an isomorphism $\mathcal{G}_{w'k}/\mathcal{G}_{w'k}^\circ \cong \mathfrak{S}_4$.

By computation,

$$(6.3) \quad \begin{aligned} g^{-1}\overline{\tau}_1g &= \left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \right), \\ g^{-1}\overline{\tau}_2g &= \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \right), \\ g^{-1}\overline{\tau}_3g &= \left(\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \right). \end{aligned}$$

Therefore,

$$(6.4) \quad \rho(g^{-1}\overline{\tau}_1g) = \tau_1, \rho(g^{-1}\overline{\tau}_2g) = \tau_2, \rho(g^{-1}\overline{\tau}_3g) = \tau_1(-I_8, -I_8, I_8),$$

where τ_1, τ_2 are the elements in (3.10) and $(I_8, -I_8, -I_8) \in \text{Spin}(Q)$ (with the understanding that elements of $\text{Spin}(Q)$ act on diagonal entries of elements of \mathcal{J} trivially). Note that $\tau_1, \tau_2, \tau_1(-I_8, -I_8, I_8)$ correspond to $(1,2), (2,3), (1,2) \in \mathfrak{S}_3$ by the homomorphism η_w from G_w to \mathfrak{S}_3 (see (3.14) and (3.22)). These elements

generate a subgroup of G_{wk} which is isomorphic to \mathfrak{S}_4 . We identify \mathfrak{S}_4 with this subgroup.

For $x \in \mathcal{V}_k^{\text{ss}}$, let $\text{Zero}_{\mathbb{P}^2}(x) = \{p_{x,1}, \dots, p_{x,4}\}$. We define a map $\phi_x : \text{Gal}(k^{\text{sep}}/k) \rightarrow \mathfrak{S}_4$ by

$$(6.5) \quad p_{x,j}^\sigma = p_{x,\phi_x(\sigma)(j)}$$

for $\sigma \in \text{Gal}(k^{\text{sep}}/k)$ and $j = 1, \dots, 4$. It is easy to verify that ϕ_x is a homomorphism.

Suppose that $x = g_x w \in \mathcal{V}_k^{\text{ss}}$ where $g_x = (g_{x,1}, g_{x,2}) \in \mathcal{G}_{k^{\text{sep}}}$. We define another map $\psi_x : \text{Gal}(k^{\text{sep}}/k) \rightarrow \mathfrak{S}_4$ by

$$(6.6) \quad \psi_x(\sigma) = \gamma_w(g_x^{-1} g_x^\sigma)^{-1}$$

for $\sigma \in \text{Gal}(k^{\text{sep}}/k)$. Since the map $\text{Gal}(k^{\text{sep}}/k) \ni \sigma \mapsto \gamma_w(g_x^{-1} g_x^\sigma) \in \mathfrak{S}_4$ is an anti-homomorphism, ψ_x is a homomorphism.

By changing the numbering of $\text{Zero}_{\mathbb{P}^2}(x)$, we may assume that $p_{x,j} = p_{w,j} g_{x,1}^{-1}$ ($j = 1, \dots, 4$). Then

$$p_{x,j}^\sigma = p_{w,j}(g_{x,1}^\sigma)^{-1} = p_{w,j}(g_{x,1}^\sigma)^{-1} g_{x,1} g_{x,1}^{-1} = p_{w,\psi_w(\sigma)(j)} g_{x,1}^{-1} = p_{x,\psi_w(\sigma)(j)}.$$

Therefore, ϕ_x and ψ_x are conjugate. Hence, if we identify $H^1(k, \mathfrak{S}_4)$ with conjugacy classes of homomorphisms from $\text{Gal}(k^{\text{sep}}/k)$ to \mathfrak{S}_4 , then the action of $\text{Gal}(k^{\text{sep}}/k)$ on $\{1, 2, 3, 4\}$ can be identified with the action of $\text{Gal}(k^{\text{sep}}/k)$ on $\text{Zero}_{\mathbb{P}^2}(x)$ up to conjugation.

The following theorem is proved in [14, p.310, Theorem 5.3] (see [16, p.120, Lemma (1.8)] also).

Theorem 6.7. *The set $\mathcal{G}_k \setminus \mathcal{V}_k^{\text{ss}}$ is in bijective correspondence with $H^1(k, \mathfrak{S}_4)$. If $x \in \mathcal{V}_k^{\text{ss}}$, then ϕ_x is the corresponding homomorphism from $\text{Gal}(k^{\text{sep}}/k)$ to \mathfrak{S}_4 .*

We next review known results regarding rational orbits of the space of binary cubic forms.

Let \mathcal{W} be the space of cubic forms in two variables $v = (v_1, v_2)$ and $\mathcal{H} = \text{GL}(1) \times \text{GL}(2)$. Since we are assuming that $\text{ch}(k) \neq 3$, \mathcal{W} is an irreducible representation of \mathcal{H} where $\alpha \in \text{GL}(1)$ acts by multiplication by α . It is well-known and in fact very easy to prove that $(\mathcal{H}, \mathcal{W})$ is a regular prehomogeneous vector space, $w'' = F_w(v) = v_1 v_2 (v_1 - v_2) \in \mathcal{W}_k^{\text{ss}}$, $\mathcal{H}_{w''}/\mathcal{H}_{w''}^\circ \cong \mathfrak{S}_3$ and the set $\mathcal{H}_k \setminus \mathcal{W}_k^{\text{ss}}$ is in bijective correspondence with $H^1(k, \mathfrak{S}_3)$. Moreover, $x \in \mathcal{W}$ is semi-stable if and only if x has three distinct factors.

If $x = x(v) \in \mathcal{W}_k^{\text{ss}}$, then we define

$$(6.8) \quad \text{Zero}_{\mathbb{P}^1}(x) = \{q \in \mathbb{P}_k^1 \mid x(q) = 0\}$$

and call it the zero set of x . Obviously, if $x \in \mathcal{V}_k^{\text{ss}}$, then $\text{Zero}(x) = \text{Zero}_{\mathbb{P}^1}(F_x)$. If $g = (g_1, g_2) \in \mathcal{H}_{\bar{k}}$, $x, y \in \mathcal{W}_{\bar{k}}^{\text{ss}}$ and $x = gy$, then it is easy to see that $\text{Zero}_{\mathbb{P}^1}(x) = \text{Zero}_{\mathbb{P}^1}(y) g_2^{-1}$. In particular, if $g_x w'' = x$ where $g_x = (g_{x,1}, g_{x,2}) \in \mathcal{H}_{\bar{k}}$, then $\text{Zero}_{\mathbb{P}^1}(x) = \text{Zero}_{\mathbb{P}^1}(w'') g_{x,2}^{-1}$. If $\alpha \in \bar{k}$ and $(\alpha, 1) \in \text{Zero}_{\mathbb{P}^1}(x)$, then we may call α a root of x also.

Let $\text{Zero}_{\mathbb{P}^1}(x) = \{q_{x,1}, q_{x,2}, q_{x,3}\}$. For $g = (g_1, g_2) \in \mathcal{H}$, let $\theta_x(g) \in \mathfrak{S}_3$ be the element such that

$$q_{x,i}g_2 = q_{x,\theta_x(g)(i)}$$

for $i = 1, 2, 3$. We define a map θ_x by

$$\theta_x : \mathcal{H} \ni g \mapsto \theta_x(g) \in \mathfrak{S}_3.$$

By the same argument as in (3.15), θ_x is a homomorphism.

If $x = g_x w'' \in \mathcal{W}_k^{\text{ss}}$ ($g_x \in \mathcal{H}_{w'' k^{\text{sep}}}$), then $\text{Gal}(k^{\text{sep}}/k) \ni \sigma \mapsto \theta_{w''}(g_x^{-1} g_x^\sigma)$ is the 1-cocycle which defines the element of $H^1(k, \mathfrak{S}_3)$ which corresponds to x . Let $\lambda_x(\sigma) = \theta_{w''}(g_x^{-1} g_x^\sigma)^{-1}$ for $\sigma \in \text{Gal}(k^{\text{sep}}/k)$. Then λ_x is a homomorphism from $\text{Gal}(k^{\text{sep}}/k)$ to \mathfrak{S}_3 .]

We define another map $\kappa_x : \text{Gal}(k^{\text{sep}}/k) \rightarrow \mathfrak{S}_3$ by

$$(6.9) \quad q_{x,i}^\sigma = q_{x,\kappa_x(\sigma)(i)}$$

for $\sigma \in \text{Gal}(k^{\text{sep}}/k)$ and $i = 1, 2, 3$. Then κ_x is also a homomorphism and as in the case of \mathcal{V} , κ_x and λ_x are conjugate. So when we consider conjugacy classes, κ_x and λ_x can be identified.

We now start the proof of Theorem 1.19.

Proof of Theorem 1.19. Let $h_{\eta_w}^1 : H^1(k, G_w) \rightarrow H^1(k, \mathfrak{S}_3)$ be the map induced by η_w . This of course induces a map from the subset $\text{Ker}(H^1(k, G_w) \rightarrow H^1(k, G))$ of $H^1(k, G_w)$ to $H^1(k, \mathfrak{S}_3)$. We first prove that if this map from $\text{Ker}(H^1(k, G_w) \rightarrow H^1(k, G))$ to $H^1(k, \mathfrak{S}_3)$ is a bijection, then the element of $H^1(k, \mathfrak{S}_3)$ which corresponds to the orbit of $x \in V_k^{\text{ss}}$ is the one obtained by the action of $\text{Gal}(k^{\text{sep}}/k)$ on the set $\text{Zero}(x)$. Note that the set $\text{Ker}(H^1(k, G_w) \rightarrow H^1(k, G))$ is in bijective correspondence with $G_k \backslash V_k^{\text{ss}}$.

We consider the homomorphism $G \rightarrow \mathcal{H}$ defined as follows:

$$\Phi : G \ni (g_1, g_2) \mapsto (c(g_1), g_2) \in \mathcal{H}.$$

It is easy to see that for any $x \in V$ and $g \in G$, $F_{gx} = \Phi(g)(F_x)$. Since $F_w = w''$, $\Phi(G_w) \subset \mathcal{H}_{w''}$. By choosing $q_{w'',1} = (0, 1)$, $q_{w'',2} = (1, 1)$, $q_{w'',3} = (1, 0)$, the definitions of $\eta_w, \theta_{w''}$ imply that the following diagram

$$(6.10) \quad \begin{array}{ccc} G_w & \xrightarrow{\eta_w} & \mathfrak{S}_3 \\ \Phi \downarrow & \nearrow \theta_{w''} & \\ \mathcal{H}_{w''} & & \end{array}$$

is commutative.

Let $x = g_x w \in V_k^{\text{ss}}$ where $g_x \in G_{k^{\text{sep}}}$. Then $F_x = \Phi(g_x)w''$. Let c_x be the element of $H^1(k, G_w)$ as in (1.14). Then F_x corresponds to the cohomology class in $H^1(k, \mathcal{H}_{w''})$ defined by the 1-cocycle

$$\text{Gal}(k^{\text{sep}}/k) \ni \sigma \mapsto \Phi(g_x)^{-1} \Phi(g_x)^\sigma = \Phi(g_x^{-1} g_x^\sigma) \in \mathcal{H}_{w'' k^{\text{sep}}}.$$

We define $h_\Phi^1 : H^1(k, G_w) \rightarrow H^1(k, \mathcal{H}_{w''})$, $h_{\theta_{w''}}^1 : H^1(k, \mathcal{H}_{w''}) \rightarrow H^1(k, \mathfrak{S}_3)$ similarly as above. Then by the commutativity of the above diagram,

$$h_{\eta_w}^1(c_x)^{-1} = h_{\theta_{w''}}^1(h_\Phi^1(c_x))^{-1} = \lambda_{F_x} = \kappa_{F_x}.$$

Since the correspondence $x \mapsto h_{\eta_w}^1(c_x)^{-1}$ is bijective, so is the correspondence $x \mapsto \kappa_{F_x}$. So the orbits in $G_k \backslash V_k^{\text{ss}}$ are determined by the action of the Galois group on $\text{Zero}(x)$.

To prove the above bijectivity, we first consider the case where k is a finite field. In this case, the well-known theorem of Lang [8] says that the first Galois cohomology set of any smooth connected algebraic group is trivial. So $H^1(k, G) = \{1\}$. Therefore, $G_k \backslash V_k^{\text{ss}}$ is in bijective correspondence with $H^1(k, G_w)$. The map ϕ in the exact sequence (3.17) can be replaced by η_w by the commutativity of the diagram (3.16). By [16, p.120, Lemma (1.8)], $h_{\eta_w}^1$ is injective. Since the sequence (3.17) is split, it is surjective. Hence, the map $h_{\eta_w}^1$ is bijective. This completes the proof of Theorem 1.19 for the case where k is a finite field.

Therefore, we assume for the rest of this section that k is an infinite field and \mathcal{O} is the split octonion $\tilde{\mathcal{O}}$. The reason why we can prove the theorem when the octonion is split is that the norm map $\mathcal{O}^\times \rightarrow k^\times$ is surjective. Note that \mathcal{O}^\times contains $\text{GL}(2)_k$ as a subset, the restriction of the norm of \mathcal{O}^\times to $\text{GL}(2)_k$ coincides with the determinant and the determinant induces a surjective homomorphism $\text{GL}(2)_k \rightarrow k^\times$ ($\text{GL}(2)_k$ is a group).

What we need for later consideration about the space \mathcal{V} is the following. First, the stabilizer of 4 in \mathfrak{S}_4 is obviously isomorphic to \mathfrak{S}_3 . Since this obvious subgroup plays an important role in the subsequent proof, we name this subgroup as follows.

Definition 6.11. $\mathcal{S} = \mathfrak{S}_3 \subset \mathfrak{S}_4$.

As we pointed out after (6.4) that \mathcal{S} is generated by $g^{-1}\overline{\tau_1}g$, $g^{-1}\overline{\tau_2}g$ of (6.3). As elements of \mathfrak{S}_4 , these elements are the transpositions (12), (23). The equation (3.22) implies that (12), (23) $\in \mathcal{S}$ map to (12), (23) $\in \mathfrak{S}_3$ by η_w . Therefore, η_w induces an isomorphism $\mathcal{S} \cong \mathfrak{S}_3$.

We shall prove the following proposition.

Proposition 6.12. *Any orbit in $G_k \backslash V_k^{\text{ss}}$ has a representative in $\mathcal{V}_k^{\text{ss}}$. Moreover, such a representative can be chosen so that it corresponds to a cohomology class in $H^1(k, \mathfrak{S}_4)$ which comes from $H^1(k, \mathcal{S})$.*

We show that Theorem 1.19 now follows from Proposition 6.12. Suppose that Proposition 6.12 holds.

The natural maps $\mathcal{S} \rightarrow \mathfrak{S}_4 \rightarrow \mathcal{G}_w \rightarrow G_w \rightarrow \mathcal{H}_{w''} \rightarrow \mathfrak{S}_3$ induce maps:

$$H^1(k, \mathcal{S}) \rightarrow H^1(k, \mathfrak{S}_4) \rightarrow H^1(k, \mathcal{G}_w) \rightarrow H^1(k, G_w) \rightarrow H^1(k, \mathcal{H}_{w''}) \rightarrow H^1(k, \mathfrak{S}_3).$$

Note that the homomorphism $G_w \rightarrow \mathfrak{S}_3$ can be regarded as η_w . Since the map $H^1(k, \mathcal{G}_w) \rightarrow H^1(k, \mathcal{G}_w / \mathcal{G}_w^\circ) \cong H^1(k, \mathfrak{S}_4)$ is bijective (\cong means bijection) and $\mathfrak{S}_4 \rightarrow \mathcal{G}_w$ is a section, the map $H^1(k, \mathfrak{S}_4) \rightarrow H^1(k, \mathcal{G}_w)$ is bijective. The map $H^1(k, \mathcal{G}_w) \rightarrow H^1(k, G_w)$ induces a map

$$H^1(k, \mathcal{G}_w) \cong \text{Ker}(H^1(k, \mathcal{G}_w) \rightarrow H^1(k, \mathcal{G})) \rightarrow \text{Ker}(H^1(k, G_w) \rightarrow H^1(k, G)).$$

By assumption, the map

$$(6.13) \quad H^1(k, \mathcal{S}) \rightarrow \text{Ker}(H^1(k, G_w) \rightarrow H^1(k, G)) \cong G_k \backslash V_k^{\text{ss}}$$

is surjective. Since the composition

$$H^1(k, \mathcal{S}) \rightarrow H^1(k, G_w) \rightarrow H^1(k, \mathfrak{S}_3)$$

is bijective, (6.13) is injective and so bijective. Since $H^1(k, \mathcal{S}) \rightarrow H^1(k, \mathfrak{S}_3)$ is bijective,

$$\text{Ker}(H^1(k, G_w) \rightarrow H^1(k, G)) \rightarrow H^1(k, \mathfrak{S}_3)$$

is bijective and this map is induced by η_w .

We now prove Proposition 6.12.

Proof of Proposition 6.12. If $x \in \mathcal{V}_k^{\text{ss}}$ and $\text{Zero}_{\mathbb{P}^2}(x)$ has a k -rational point, then the element of $H^1(k, \mathfrak{S}_4)$ which corresponds to x comes from $H^1(k, \mathcal{S})$. So what we have to prove is that any orbit in $G_k \backslash V_k^{\text{ss}}$ has a representative $x \in \mathcal{V}_k^{\text{ss}}$ such that $\text{Zero}_{\mathbb{P}^2}(x)$ has a k -rational point.

We first describe some elements of G_1 . This will be needed when we show later that any orbit in $G_k \backslash V_k^{\text{ss}}$ has a representative in $\mathcal{V}_k^{\text{ss}}$.

For $\alpha \in \mathcal{O}^\times$, let $d_1(\alpha), d_2(\alpha), d_3(\alpha)$ be the k -linear maps $\mathcal{J} \rightarrow \mathcal{J}$ defined by

$$(6.14) \quad \begin{aligned} d_1(\alpha) : \mathcal{J} \ni \begin{pmatrix} s_1 & x_3 & \overline{x_2} \\ \overline{x_3} & s_2 & x_1 \\ x_2 & \overline{x_1} & s_3 \end{pmatrix} &\mapsto \begin{pmatrix} \|\alpha\|s_1 & x_3\overline{\alpha} & \overline{x_2}\alpha \\ \alpha\overline{x_3} & s_2 & \alpha x_1\alpha/\|\alpha\| \\ \overline{\alpha}x_2 & \overline{\alpha}x_1\overline{\alpha}/\|\alpha\| & s_3 \end{pmatrix} \in \mathcal{J}, \\ d_2(\alpha) : \mathcal{J} \ni \begin{pmatrix} s_1 & x_3 & \overline{x_2} \\ \overline{x_3} & s_2 & x_1 \\ x_2 & \overline{x_1} & s_3 \end{pmatrix} &\mapsto \begin{pmatrix} s_1 & \overline{\alpha}x_3 & \overline{\alpha}x_2\overline{\alpha}/\|\alpha\| \\ \overline{x_3}\alpha & \|\alpha\|s_2 & x_1\overline{\alpha} \\ \alpha x_2\alpha/\|\alpha\| & \alpha\overline{x_1} & s_3 \end{pmatrix} \in \mathcal{J}, \\ d_3(\alpha) : \mathcal{J} \ni \begin{pmatrix} s_1 & x_3 & \overline{x_2} \\ \overline{x_3} & s_2 & x_1 \\ x_2 & \overline{x_1} & s_3 \end{pmatrix} &\mapsto \begin{pmatrix} s_1 & \alpha x_3\alpha/\|\alpha\| & \alpha\overline{x_2} \\ \overline{\alpha}x_3\overline{\alpha}/\|\alpha\| & s_2 & \overline{\alpha}x_1 \\ x_2\overline{\alpha} & \overline{x_1}\alpha & \|\alpha\|s_3 \end{pmatrix} \in \mathcal{J}. \end{aligned}$$

Also for $\alpha_1, \alpha_2, \alpha_3 \in k^\times$, let

$$(6.15) \quad d(\alpha_1, \alpha_2, \alpha_3) = \rho_1(\text{diag}(\alpha_1, \alpha_2, \alpha_3)) \in \text{GL}(\mathcal{J}).$$

For $u \in \mathcal{O}$, let $n_{21}(u) : \mathcal{J} \rightarrow \mathcal{J}$ be the k -linear map defined by

$$(6.16) \quad \mathcal{J} \ni X \mapsto \begin{pmatrix} 1 & & \\ u & 1 & \\ & & 1 \end{pmatrix} X \begin{pmatrix} 1 & \overline{u} & \\ & 1 & \\ & & 1 \end{pmatrix} \in \mathcal{J}.$$

We define $n_{ij}(u)$ for $i \neq j$ similarly.

Lemma 6.17. (1) For all $\alpha_1, \alpha_2, \alpha_3 \in k^\times$ and $u \in \mathcal{O}$, the right hand sides of (6.15), (6.16) do not depend on the order of multiplication and the resulting elements belong to \mathcal{J} .

(2) For all $\alpha \in \mathcal{O}^\times$, the maps $d_i(\alpha)$ ($i = 1, 2, 3$) belong to $G_{1k} = GE_{6k}$. Moreover, if $\|\alpha\| = 1$, then the maps $d_i(\alpha)$ ($i = 1, 2, 3$) belong to $H_{1k} = E_{6k}$.

(3) If $\alpha_1, \alpha_2, \alpha_3 \in k^\times$, then the map $d(\alpha_1, \alpha_2, \alpha_3)$ belong to $G_{1k} = GE_{6k}$. Moreover, $c(d(\alpha_1, \alpha_2, \alpha_3)) = (\alpha_1\alpha_2\alpha_3)^2$.

(4) The maps $n_{ij}(u)$ ($i \neq j$), belong to $H_{1k} = E_{6k}$.

Proof. The proof is easy for $d(\alpha_1, \alpha_2, \alpha_3)$. We only consider $d_1(\alpha)$ and $n_{21}(u)$ since other cases are similar.

Let $\alpha \in \mathcal{O}^\times$. It is known that for any $a, x, y \in \mathcal{O}$,

$$(6.18) \quad a(xy)a = (ax)(ya), \quad a(xa) = (ax)a$$

(see [12, p.9, Proposition 1.4.1], [12, p.10, Lemma 1.4.2]).

We first consider $d_1(\alpha)$ ($\alpha \in \mathcal{O}^\times$). Let X be as in (1.3). By (1.2) and (6.18),

$$\begin{aligned} \text{tr}((\alpha x_1 \alpha)(\bar{\alpha} x_2)(x_3 \bar{\alpha})) &= \text{tr}((\alpha x_1 \alpha)(\bar{\alpha}(x_2 x_3) \bar{\alpha})) = \text{tr}(((\alpha x_1 \alpha) \bar{\alpha})((x_2 x_3) \bar{\alpha})) \\ &= \text{tr}((((\alpha x_1) \alpha) \bar{\alpha})((x_2 x_3) \bar{\alpha})) = \|\alpha\| \text{tr}((\alpha x_1)((x_2 x_3) \bar{\alpha})) \\ &= \|\alpha\| \text{tr}(\bar{\alpha}(\alpha x_1)(x_2 x_3)) = \|\alpha\|^2 \text{tr}(x_1(x_2 x_3)). \end{aligned}$$

So,

$$\begin{aligned} \det(d_1(\alpha)(X)) &= \|\alpha\| s_1 s_2 s_3 + \frac{1}{\|\alpha\|} \text{tr}((\alpha x_1 \alpha)(\bar{\alpha} x_2)(x_3 \bar{\alpha})) \\ &\quad - \frac{1}{\|\alpha\|} s_1 \|\alpha x_1 \alpha\| - s_2 \|\bar{\alpha} x_2\| - s_3 \|x_3 \bar{\alpha}\| \\ &= \|\alpha\| s_1 s_2 s_3 + \|\alpha\| \text{tr}(x_1 x_2 x_3) \\ &\quad - \|\alpha\| s_1 \|x_1\| - \|\alpha\| s_2 \|x_2\| - \|\alpha\| s_3 \|x_3\| \\ &= \|\alpha\| \det(X). \end{aligned}$$

Therefore, $d_1(\alpha)$ belongs to GE_6 . Moreover, $d_1(\alpha)$ belongs to E_6 if $\|\alpha\| = 1$.

Next, we consider $n_{21}(u)$ ($u \in \mathcal{O}$). Let X be as in (1.3). By taking the product of the first two matrices first,

$$\begin{aligned} n_{21}(u)(X) &= \begin{pmatrix} s_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 + s_1 u & s_2 + u x_3 & x_1 + u \bar{x}_2 \\ x_2 & \bar{x}_1 & s_3 \end{pmatrix} \begin{pmatrix} 1 & \bar{u} \\ & 1 \\ & & 1 \end{pmatrix} \\ &= \begin{pmatrix} s_1 & x_3 + s_1 \bar{u} & \bar{x}_2 \\ \bar{x}_3 + s_1 u & s_2 + \text{tr}(u x_3) + s_1 \|u\| & x_1 + u \bar{x}_2 \\ x_2 & \bar{x}_1 + x_2 \bar{u} & s_3 \end{pmatrix}. \end{aligned}$$

One can also verify that changing the order of multiplication does not change the result.

By definition,

$$\begin{aligned} \det(n_{21}(u)(X)) &= s_1(s_2 + \text{tr}(u x_3) + s_1 \|u\|) s_3 + \text{tr}(((x_1 + u \bar{x}_2) x_2)(x_3 + s_1 \bar{u})) \\ &\quad - s_1 \|x_1 + u \bar{x}_2\| - (s_2 + \text{tr}(u x_3) + s_1 \|u\|) \|x_2\| - s_3 \|x_3 + s_1 \bar{u}\| \\ &= \det(X) + s_1 s_3 \text{tr}(u x_3) + s_1^2 s_3 \|u\| \\ &\quad + s_1 \text{tr}((x_1 x_2) \bar{u}) + \text{tr}((u \bar{x}_2) x_2) x_3 + s_1 \text{tr}((u \bar{x}_2) x_2) \bar{u} \\ &\quad - s_1 \|u\| \|x_2\| - s_1 \text{tr}(x_1(x_2 \bar{u})) - (\text{tr}(u x_3) + s_1 \|u\|) \|x_2\| \\ &\quad - s_1^2 s_3 \|u\| - s_1 s_3 \text{tr}(x_3 u). \end{aligned}$$

It is known that for any $x, y \in \mathcal{O}$,

$$(xy)\bar{y} = x\|y\|, \quad \bar{y}(yx) = x\|y\|.$$

(see [12, p.8, Lemma 1.3.3]). So, by this and the relations in (1.2),

$$\begin{aligned} \operatorname{tr}((x_1 x_2) \bar{u}) &= \operatorname{tr}(x_1 (x_2 \bar{u})), \\ \operatorname{tr}(((u \bar{x}_2) x_2) x_3) &= \|x_2\| \operatorname{tr}(u x_3) = \|x_2\| \operatorname{tr}(x_3 u), \\ \operatorname{tr}(((u \bar{x}_2) x_2) \bar{u}) &= \|x_2\| \operatorname{tr}(u \bar{u}) = 2\|x_2\| \|u\|. \end{aligned}$$

So all the terms except for $\det(X)$ cancel out and we obtain $\det(n_{21}(u)(X)) = \det(X)$. \square

Note that for $X \in \mathcal{J}$ and $\alpha \in \mathbb{O}^\times$, the determinant of

$$(\operatorname{diag}(\alpha, 1, 1)X) \operatorname{diag}(\bar{\alpha}, 1, 1)$$

may not be $\|\alpha\| \det(X)$.

Since $\operatorname{GL}(3)$ is generated by elements of the forms $d(\alpha_1, \alpha_2, \alpha_3)$ ($\alpha_1, \alpha_2, \alpha_3 \in k^\times$), $n_{ij}(u)$ ($i \neq j, u \in k$), the following corollary follows.

Corollary 6.19. *For any $g_1 \in \operatorname{GL}(3)$ (resp. $g \in \mathcal{G}$), $\rho(g_1) \in G_1$ (resp. $\rho(g) \in G$).*

Thus, if $x_1, x_2 \in \mathcal{V}_k^{\text{ss}}$ belong to the same orbit in $\mathcal{G} \setminus \mathcal{V}_k^{\text{ss}}$, then they belong to the same orbit in $G_k \setminus \mathcal{V}_k^{\text{ss}}$.

In the following, if $x = (x_1, x_2) \in V_k$, then we assume that

$$(6.20) \quad x_i = h(s_{i1}, s_{i2}, s_{i3}, x_{i1}, x_{i2}, x_{i3}) = \begin{pmatrix} s_{i,1} & x_{i,3} & \bar{x}_{i,2} \\ \bar{x}_{i,3} & s_{i,2} & x_{i,1} \\ x_{i,2} & \bar{x}_{i,1} & s_{i,3} \end{pmatrix}$$

where $s_{i,j} \in k, x_{i,j} \in \mathbb{O}$ for $i = 1, 2, j = 1, 2, 3$.

Lemma 6.21. *Suppose that $x = (x_1, x_2) \in V_k^{\text{ss}}$ and that $\det(x_1) = 0$. Then one can choose $g \in G_k$ so that after replacing x by gx , x is in the form:*

$$(6.22) \quad \left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & s_{2,2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right).$$

Proof. We first show that there exists $g \in G_1$ such that the $(1,1)$ -entry of gx_1 is non-zero.

Let W' be the subgroup of $\operatorname{GL}(3)_k$ consisting of permutation matrices. Then W' is isomorphic to \mathfrak{S}_3 . Let $W = \rho_1(W') \subset G_1$. We put

$$(6.23) \quad \nu' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \nu = \rho_1(\nu').$$

Then $\nu' \in W', \nu \in W$.

If there exists $1 \leq j \leq 3$ such that $s_{1,j} \neq 0$, then by applying an element of W , we may assume that $s_{1,1} \neq 0$. Suppose that all diagonal entries of x_1 are 0. Since $x \in V_k^{\text{ss}}, x_1 \neq 0$. So there exists j such that $x_{1,j} \neq 0$. By applying an element of W , we may assume that $x_{1,3} \neq 0$. If $u \in \mathbb{O}$, then the $(1,1)$ -entry of $n_{12}(u)(x_1)$ is $\operatorname{tr}(u \bar{x}_{1,3})$. Since $x_{1,3} \neq 0$, there exists $u \in \mathbb{O}$ such that $\operatorname{tr}(u \bar{x}_{1,3}) \neq 0$. Therefore, we may assume that $s_{1,1} \neq 0$.

By applying elements of the forms $n_{21}(u), n_{31}(u)$ ($u \in \mathcal{O}$), we may assume that $x_{1,2} = x_{1,3} = 0$. If $s_{1,2} = s_{1,3} = x_{1,1} = 0$, then the weights of all non-zero coordinates of x with respect to the one parameter subgroup (which will be abbreviated as 1PS from now on)

$$\lambda : \mathrm{GL}(1) \ni \alpha \mapsto (d(\alpha^2, \alpha^{-1}, \alpha^{-1}), \mathrm{diag}(\alpha^{-3}, \alpha^3)) \in G_k$$

is positive (this $\mathrm{GL}(1)$ is $\mathrm{GL}(1)$ over k). This contradicts to the assumption that $x \in V_k^{\mathrm{ss}}$. This is well-known (see [9, p.49, Theorem 2.1]) but can be seen in this situation as follows also. If $\Delta(x)$ is the relative invariant polynomial of degree 12 as in Introduction, since $\mathrm{Im}(\lambda) \subset E_6 \times \mathrm{SL}(2)$, $\Delta(\lambda(\alpha)x) = \Delta(x)$. However, since the weights of all non-zero coordinates with respect to λ are positive, there exists a polynomial $P(x, \alpha)$ such that $\Delta(x) = \Delta(\lambda(\alpha)x) = \alpha P(x, \alpha)$. Since $\Delta(x)$ is a non-zero polynomial which does not depend on α , this is a contradiction.

So $(s_{1,2}, s_{1,3}, x_{1,1}) \neq (0, 0, 0)$. By a similar argument as above, there exists $g \in G_{1k}$ such that gx_1 is in the form $\mathrm{diag}(s_{1,1}, s_{1,2}, s_{1,3})$ where $s_{1,1}, s_{1,2} \neq 0$. Since $\det(x_1) = 0$, $s_{1,3} = 0$.

If $s_{2,3} = 0$ then the weights of all non-zero coordinates with respect to the 1PS

$$\lambda : \mathrm{GL}(1) \ni \alpha \mapsto (d(\alpha^2, \alpha^2, \alpha^{-4}), \mathrm{diag}(\alpha^{-3}, \alpha^3))$$

is positive, which is a contradiction. Therefore, $s_{2,3} \neq 0$.

Then by applying elements of the forms $n_{31}(u), n_{32}(u)$, we may assume that $x_{2,1} = x_{2,2} = 0$. Note that $n_{31}(u), n_{32}(u)$ do not change x_1 . Therefore, x is in the form:

$$(6.24) \quad \left(\begin{pmatrix} s_{1,1} & & \\ & s_{1,2} & \\ & & 0 \end{pmatrix}, \begin{pmatrix} \frac{s_{2,1}}{\bar{x}_{2,3}} & x_{2,3} & 0 \\ & s_{2,2} & 0 \\ 0 & 0 & s_{2,3} \end{pmatrix} \right).$$

Since the norm map $\mathcal{O}^\times \rightarrow k^\times$ is surjective, by applying elements of the forms $d_1(\alpha_1), d_2(\alpha_2), d_3(\alpha_3)$ ($\alpha_1, \alpha_2, \alpha_3 \in \mathcal{O}^\times$), we may assume that $s_{1,1} = 1/2, s_{1,2} = -1/2, s_{2,3} = 1$. Since

$$(6.25) \quad \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we may assume that x is in the form:

$$(6.26) \quad x = (x_1, x_2) = \left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{s_{2,1}}{\bar{x}_{2,3}} & x_{2,3} & 0 \\ & s_{2,2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right).$$

We first show that there exists $g \in G_k$ such that gx is still in the form (6.26) and that $s_{2,1} \neq 0$.

Suppose that $s_{2,1} = 0$. If $s_{2,2} \neq 0$, then we only have to apply an element of W . So we assume that $s_{2,1} = s_{2,2} = 0$. In this situation, if $u \in \mathcal{O}$ and $\mathrm{tr}(u) = 0$, then $n_{12}(u)x_1 = x_1$ and the $(1,1)$ -entry of $n_{12}(u)x_2$ is $\mathrm{tr}(u\bar{x}_{2,3}) = \mathrm{tr}(\bar{x}_{2,3}u)$. Let $\bar{x}_{2,3} = a + b$ where $a \in k$ and $\mathrm{tr}(b) = 0$. Then $\mathrm{tr}(\bar{x}_{2,3}u) = \mathrm{tr}(bu)$. The assumption implies that $b \neq 0$. Suppose that $\mathrm{tr}(bu) = 0$ for all $u \in \mathcal{O}$ such that $\mathrm{tr}(u) = 0$. Since

$\text{tr}(bc) = 0$ if $c \in k$, $\text{tr}(bu) = 0$ for all $u \in \mathcal{O}$. Since $\mathcal{O}^2 \ni (u_1, u_2) \mapsto \text{tr}(u_1 u_2)$ is a non-degenerate bilinear form, $b = 0$, which is a contradiction. Therefore, there exists $u \in \mathcal{O}$ such that $\text{tr}(u) = 0$ and that $\text{tr}(\overline{x_{23}}u) \neq 0$. So we may assume that $s_{2,1} \neq 0$.

We next show that there exists $g \in G_k$ such that gx is still in the form (6.26) and that $x_{2,3} \in k$.

Lemma 6.27. *Suppose that $s \in k^\times$ and $x \in \mathcal{O}$. Then there exists $u \in \mathcal{O}$ such that $\text{tr}(u) = 0$ and $x + su \in k$.*

Proof. It is enough to take $u = (1/2s)(\overline{x} - x)$. \square

If $\text{tr}(u) = 0$, then $n_{12}(u)x_1 = x_1$ and the $(1,2)$ -entry of $n_{12}(u)x_2$ is $x_{2,3} + s_{2,1}u$. Since $s_{2,1} \neq 0$, there exists u such that $\text{tr}(u) = 0$ and $x_{2,3} + s_{2,1}u \in k$ by Lemma 6.27. So we may assume further that $x_{2,3} \in k$. Then if we apply the element $\begin{pmatrix} 1 & 0 \\ -x_{2,3} & 1 \end{pmatrix} \in \text{GL}(2)_k$, then $x_{2,3}$ changes to 0.

By applying $d_2(\alpha_2)d_1(\alpha_1)$, x changes to

$$\left(\begin{pmatrix} 0 & \overline{\alpha_2} \overline{\alpha_1} & 0 \\ \alpha_1 \alpha_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} s_{2,1} \|\alpha_1\| & 0 & 0 \\ 0 & s_{2,2} \|\alpha_2\| & 0 \\ 0 & 0 & 1 \end{pmatrix} \right).$$

We can choose α_1, α_2 so that $s_1 \|\alpha_1\| = -1$ and $\alpha_1 \alpha_2 = 1$. Then x changes to

$$\left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & s_{2,2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right).$$

This completes the proof of the lemma. \square

Suppose that $F_x(v)$ has a rational factor. By applying an element of $\text{GL}(2)_k$, we may assume that $F_x(1,0) = 0$. Then $\det(x_1) = 0$. By Lemma 6.21, we may assume that $x \in \mathcal{V}_k^{\text{ss}}$ and is in the form (6.22). Then $(1,0,1), (-1,0,1) \in \text{Zero}_{\mathbb{P}^2}(x)$ are rational points. Therefore, the element of $H^1(k, \mathfrak{S}_4)$ which corresponds to x with respect to the action of \mathcal{G}_k comes from $H^1(k, \mathfrak{S}_2)$ and so comes from $H^1(k, \mathcal{S})$.

Finally we consider the case where F_x is irreducible. We assume that x_1, x_2 are in the form (6.20). By assumption, $\det(x_1) \neq 0$. By the same argument as in the proof of Lemma 6.21, we may assume that x_1 is a diagonal matrix. Since $\det(x_1) \neq 0$, all diagonal entries are non-zero. Also since the norm map $\mathcal{O}^\times \rightarrow k^\times$ is surjective, we may assume that $x_1 = \text{diag}(\frac{1}{2}, -2, -\frac{1}{2})$. Let

$$(6.28) \quad g = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ 1 & & -1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $\rho_1(g)x_1 = \Lambda$. So we may assume that $x_1 = \Lambda$.

For $u \in \mathcal{O}$, let

$$A_1(u) = \begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ \frac{1}{4}\|u\| & \frac{1}{2}\overline{u} & 1 \end{pmatrix}, \quad A_2(u) = \begin{pmatrix} 1 & \frac{1}{2}\overline{u} & \frac{1}{4}\|u\| \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix}.$$

Then one can easily verify that $(A_i(u)\Lambda)^t \overline{A_i(u)} = A_i(u)(\Lambda^t \overline{A_i(u)}) = \Lambda$ ($i = 1, 2$).

Lemma 6.29. (1) For $Y \in \mathcal{J}$, $(A_i(u)Y)^t \overline{A_i(u)} = A_i(u)(Y^t \overline{A_i(u)}) \in \mathcal{J}$ for $i = 1, 2$.
 (2) If we put $B_i(u)(Y) = A_i(u)Y^t \overline{A_i(u)}$, then $B_i(u) : \mathcal{J} \rightarrow \mathcal{J}$ is a k -linear map which belongs to E_6 for $i = 1, 2$.

Proof. Since the argument is similar, we only consider $A_1(u)$.

(1) Suppose that $Y = h(t_1, t_2, t_3, y_1, y_2, y_3)$ where $t_1, t_2, t_3 \in k, y_1, y_2, y_3 \in \mathbb{O}$. Then

$$\begin{aligned} & (A_1(u)Y)^t \overline{A_1(u)} \\ &= \begin{pmatrix} t_1 & y_3 & \overline{y_2} \\ \overline{y_3} + t_1 u & t_2 + u y_3 & y_1 + u \overline{y_2} \\ y_2 + \frac{1}{2} \overline{u} \overline{y_3} + \frac{1}{4} t_1 \|u\| & \overline{y_1} + \frac{1}{2} t_2 \overline{u} + \frac{1}{4} \|u\| y_3 & t_3 + \frac{1}{2} \overline{u} y_1 + \frac{1}{4} \|u\| \overline{y_2} \end{pmatrix} \\ & \quad \times \begin{pmatrix} 1 & \overline{u} & \frac{1}{4} \|u\| \\ 0 & 1 & \frac{1}{2} u \\ 0 & 0 & 1 \end{pmatrix} \quad (\times \text{ is the matrix multiplication}) \\ &= (A_{1,ij}) \end{aligned}$$

where

$$\begin{aligned} A_{1,11} &= t_1, \\ A_{1,12} &= y_3 + t_1 \overline{u}, \\ A_{1,13} &= \overline{y_2} + \frac{1}{2} y_3 u + \frac{1}{4} t_1 \|u\|, \\ A_{1,21} &= \overline{y_3} + t_1 u, \\ A_{1,22} &= t_2 + \text{tr}(u y_3) + t_1 \|u\|, \\ A_{1,23} &= y_1 + u \overline{y_2} + \frac{1}{2} t_2 u + \frac{1}{2} (u y_3) u + \frac{1}{4} \overline{y_3} \|u\| + \frac{1}{4} t_1 \|u\| u, \\ A_{1,31} &= y_2 + \frac{1}{2} \overline{u} \overline{y_3} + \frac{1}{4} t_1 \|u\|, \\ A_{1,32} &= \overline{y_1} + y_2 \overline{u} + \frac{1}{2} t_2 \overline{u} + \frac{1}{2} (\overline{u} \overline{y_3}) \overline{u} + \frac{1}{4} y_3 \|u\| + \frac{1}{4} t_1 \|u\| \overline{u}, \\ A_{1,33} &= t_3 + \frac{1}{2} \text{tr}(\overline{y_1} u) + \frac{1}{4} t_2 \|u\| + \frac{1}{4} \text{tr}(y_2) \|u\| + \frac{1}{8} \text{tr}(y_3 u) \|u\| + \frac{1}{16} t_1 \|u\|^2. \end{aligned}$$

The only places where the order of multiplication matters are $\frac{1}{2}(u y_3)u$ in A_{23} and $\frac{1}{2}(\overline{u} \overline{y_3})\overline{u}$ in A_{32} . However, by (6.18), these are equal to $\frac{1}{2}u(y_3 u)$, $\frac{1}{2}\overline{u}(\overline{y_3} \overline{u})$ respectively. So $(A_1(u)Y)^t \overline{A_1(u)} = A_1(u)(Y^t \overline{A_1(u)})$. Also,

$$(6.30) \quad \overline{(u y_3) u} = \overline{u}(\overline{y_3} \overline{u}) = (\overline{u} \overline{y_3}) \overline{u}.$$

Therefore, $\overline{A_{1,23}} = A_{1,32}$. The conditions $\overline{A_{1,12}} = A_{1,21}$, $\overline{A_{1,13}} = A_{1,31}$ and $A_{1,ii} \in k$ ($i = 1, 2, 3$) are obvious. Thus $B_1(u)(Y) \in \mathcal{J}$.

The consideration is similar for $B_2(u)(Y)$, but we list entries of $B_2(u)(Y)$ since they will be needed later. Let $B_2(u)(Y) = (A_{2,ij})$. Then

$$\begin{aligned} A_{2,11} &= t_1 + \frac{1}{2}\text{tr}(y_3u) + \frac{1}{4}t_2\|u\| + \frac{1}{4}\text{tr}(y_2)\|u\| + \frac{1}{8}\text{tr}(\overline{y_1}u)\|u\| + \frac{1}{16}t_3\|u\|^2, \\ A_{2,12} &= y_3 + \overline{y_2}\overline{u} + \frac{1}{2}t_2\overline{u} + \frac{1}{2}(\overline{u}y_1)\overline{u} + \frac{1}{4}\overline{y_1}\|u\| + \frac{1}{4}t_3\|u\|\overline{u}, \\ A_{2,13} &= \overline{y_2} + \frac{1}{2}\overline{u}y_1 + \frac{1}{4}t_3\|u\|, \\ A_{2,22} &= t_2 + \text{tr}(u\overline{y_1}) + t_3\|u\|, \\ A_{2,23} &= y_1 + t_3u, \\ A_{2,33} &= t_3, \\ A_{2,21} &= \overline{A_{2,12}}, A_{2,31} = \overline{A_{2,13}}, A_{2,32} = \overline{A_{2,23}}. \end{aligned}$$

(2) By easy computations,

$$A_1(u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{4}\|u\| & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2}\overline{u} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where the order of multiplication does not matter. One can verify by tedious computations using (6.30) that $B_1(u) = n_{31}(-\frac{1}{4}\|u\|)n_{32}(\frac{1}{2}\overline{u})n_{21}(u)$ (this is not trivial since \mathbb{O} is not associative). Therefore, $B_1(u)$ belongs to G_{1k} . \square

We successively apply elements of G_k to modify x .

We eventually want to make $s_{2,1} = 0$ and $\|x_{2,3}\| \neq 0$. However, for that purpose, we first make $s_{2,1}$ non-zero to eliminate some entries of x_2 . If $s_{2,1} = 0$ and $s_{2,3} \neq 0$, then we only have to apply ν (see (6.23)). Note that $\nu x_1 = x_1$. Suppose that $s_{2,1} = s_{2,3} = 0$. If further $x_{2,3} = 0$, then by applying $\begin{pmatrix} 1 & 0 \\ \frac{1}{2}s_{2,2} & 1 \end{pmatrix} \in \text{GL}(2)_k$, $s_{2,2}$ becomes 0. Then $\det(x_2) = 0$, which is a contradiction. Therefore, $x_{2,3} \neq 0$. If we choose $u \in \mathbb{O}$ such that $\|u\| = 0$, then by applying $B_2(u)$ to x_2 , $s_{2,1}$ (which is 0 now) becomes $\frac{1}{2}\text{tr}(x_{2,3}u)$ and x_1 does not change.

We can choose a basis of \mathbb{O} as a k -vector space so that according to the coordinate $u = (u_1, \dots, u_8)$ of u with respect to this basis, $\|u\| = u_1u_2 + u_3u_4 + u_5u_6 + u_7u_8$ (\mathbb{O} is split). Since $\frac{1}{2}\text{tr}(x_{2,3}u)$ is a non-zero k -linear function, it is in the form $c_1u_1 + \dots + c_8u_8$ ($c_1, \dots, c_8 \in k$). We may assume that $c_1 \neq 0$. We choose $u_2 = 1$. Then $\|u\| = 0$ if and only if $u_1 = -(u_3u_4 + u_5u_6 + u_7u_8)$ and so

$$\frac{1}{2}\text{tr}(x_{2,3}u) = -c_1(u_3u_4 + u_5u_6 + u_7u_8) + c_2 + c_3u_3 + \dots + c_8u_8.$$

This is a non-zero polynomial. Since k is an infinite field, there exists $u \in \mathbb{O}$ such that $\|u\| = 0$ and that $\frac{1}{2}\text{tr}(x_{2,3}u) \neq 0$. Therefore, we may assume that $s_{2,1} \neq 0$ for the moment (we later change $s_{2,1}$ to 0 after eliminating some entries of x_2).

By applying an element of the form $B_1(u)$ ($u \in \mathbb{O}$), x_1 does not change and $x_{2,3}$ changes to $x_{2,3} + s_{2,1}\overline{u}$. Since $s_{2,1} \neq 0$, there exists u such that $x_{2,3} + s_{2,1}\overline{u} = 0$. Therefore, we may assume that $x_{2,3} = 0$ for the moment (we later change $x_{2,3}$ so that $\|x_{2,3}\| \neq 0$).

By applying the element $\begin{pmatrix} 1 & 0 \\ \frac{1}{2}s_{2,2} & 1 \end{pmatrix} \in \mathrm{GL}(2)_k$, we may assume that $s_{2,2} = 0$. So x is in the form:

$$\left(\begin{pmatrix} & 1 \\ -2 & \\ 1 & \end{pmatrix}, \begin{pmatrix} s_{2,1} & 0 & \overline{x}_{2,2} \\ 0 & 0 & x_{2,1} \\ x_{2,2} & \overline{x}_{2,1} & s_{2,3} \end{pmatrix} \right).$$

Then $\det(x_2) = s_{2,1}\|x_{2,1}\| \neq 0$. Therefore, $x_{2,1} \in \mathcal{O}^\times$.

We want to make $x_{2,1} \in k$ while preserving $x_1 = \Lambda$ (see (6.28)). So we prove the following lemma.

Lemma 6.31. *Let $y = h(t_1, t_2, t_3, y_1, y_2, y_3) \in \mathcal{J}$ where $t_i \in k, y_i \in \mathcal{O}$ ($i = 1, 2, 3$). If $\|y_1\| \neq 0$, then there exists $g \in G_{1\Lambda k}$ ($G_{1\Lambda}$ is the stabilizer of Λ in G_1) such that the $(2, 3)$ entry of gy is 1 and g preserves subspaces kE_i, \mathcal{O}_i (see (2.4) and (2.6)) of \mathcal{J} for $i = 1, 2, 3$.*

Proof. We construct such an element g by combining the maps $d_i(\beta)$ where $\beta \in \mathcal{O}^\times, i = 1, 2$. For $\beta \in \mathcal{O}^\times$,

$$d_1(-\|\beta\|)d_2(\beta)(\Lambda) = \begin{pmatrix} & & -\overline{\beta}^2 \\ -\beta^2 & -2\|\beta\| & \end{pmatrix}.$$

If $\mathrm{tr}(\beta) = 0$, then since $\beta = -\overline{\beta}, \beta^2 = \overline{\beta}^2 = -\|\beta\|$ and so $d_1(-\|\beta\|)d_2(\beta)(\Lambda) = \|\beta\|\Lambda$. Hence, the element $\|\beta\|^{-1}I_{27}d_1(-\|\beta\|)d_2(\beta) \in G_{1k}$ fixes Λ for any $\beta \in \mathbb{W} \cap \mathcal{O}^\times$ where \mathbb{W} is the orthogonal complement of $k \cdot 1$ in \mathcal{O} . For $\beta \in \mathcal{O}^\times$, we denote the element $\|\beta\|^{-1}I_{27}d_1(-\|\beta\|)d_2(\beta) \in G_{1k}$ by $\mathcal{D}(\beta)$. By the above argument, if $\mathrm{tr}(\beta) = 0$ ($\beta \in \mathbb{W}$), then $\mathcal{D}(\beta) \in G_{1\Lambda k}$. Since $d_j(\alpha)$ where $\alpha \in \mathcal{O}^\times, j = 1, 2$ preserves subspaces kE_i, \mathcal{O}_i of \mathcal{J} for $i = 1, 2, 3$, $\mathcal{D}(\beta)$ also preserves these subspaces. By applying $\mathcal{D}(\overline{\beta})$ where $\beta \in \mathbb{W} \cap \mathcal{O}^\times$ to y , y_1 changes to $\|\beta\|^{-1}y_1\beta$ ($d_1(-\|\overline{\beta}\|)$ does not change y_1) and so $\mathrm{tr}(y_1)$ changes to $\|\beta\|^{-1}\mathrm{tr}(y_1\beta)$.

If there exists $\beta \in \mathbb{W} \cap \mathcal{O}^\times$ such that $\mathrm{tr}(\beta y_1) = 0$, then the $(2, 3)$ entry of $\mathcal{D}(\|\beta\|^{-1}\overline{y_1\beta}) (\mathcal{D}(\overline{\beta})(y))$ is

$$\|(\|\beta\|^{-1}(\overline{y_1\beta}))\|^{-1}(\|\beta\|^{-1}(y_1\beta))(\|\beta\|^{-1}(\overline{y_1\beta})) = 1.$$

Note that by assumption, $\|\overline{y_1\beta}\| \neq 0$. Since $\mathcal{D}(\|\beta\|^{-1}\overline{y_1\beta}) \mathcal{D}(\overline{\beta})$ belongs to $G_{1\Lambda k}$ and preserves subspaces kE_i, \mathcal{O}_i of \mathcal{J} where $i = 1, 2, 3$, this element is what we wanted. So it is enough to show that there exists $\beta \in \mathbb{W} \cap \mathcal{O}^\times$ such that $\mathrm{tr}(\beta y_1) = 0$.

For $a \in \mathcal{O}$, we denote the linear map $\mathcal{O} \ni x \rightarrow \mathrm{tr}(ax) \in k$ by tr_a and denote the restriction of tr_a on \mathbb{W} by $\mathrm{tr}_a|_{\mathbb{W}}$. Then $\dim(\mathrm{Ker}(\mathrm{tr}_{y_1}|_{\mathbb{W}})) \geq 6$. Since \mathcal{O} is split, the dimension of the maximal totally isotropic subspace, which is called the Witt index, of \mathcal{O} is 4 ($= \frac{1}{2} \dim \mathcal{O}$). Thus, there exists $\beta \in \mathrm{Ker}(\mathrm{tr}_{y_1}|_{\mathbb{W}})$ such that $\|\beta\| \neq 0$. This completes the proof of the lemma. \square

By the above lemma, we may assume that x is in the form:

$$\left(\begin{pmatrix} & 1 \\ -2 & \\ 1 & \end{pmatrix}, \begin{pmatrix} s_{2,1} & 0 & \overline{x}_{2,2} \\ 0 & 0 & 1 \\ x_{2,2} & 1 & s_{2,3} \end{pmatrix} \right)$$

where $s_{2,1} \neq 0$.

If $u \in \mathcal{O}$ and $\text{tr}(u) = 0$, then $n_{31}(u)$ does not change x_1 and $x_{2,2}$ changes to $x_{2,2} + s_{2,1}u$. By Lemma 6.27, there exists u such that $\text{tr}(u) = 0$ and $x_{2,2} + s_{2,1}u \in k$. Therefore, we may assume that $x_{2,2} \in k$. By applying the element $\begin{pmatrix} 1 & 0 \\ -x_{2,2}s_{2,1}^{-1} & s_{2,1}^{-1} \end{pmatrix}$, we may assume that x is in the form:

$$\left(\begin{pmatrix} & & 1 \\ & -2 & \\ 1 & & \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & s_{2,2} & c \\ 0 & c & s_{2,3} \end{pmatrix} \right)$$

where $c \in k^\times$.

By applying $B_1(2)$ and then $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, x_1 does not change and x_2 becomes

$$x'_2 = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 6 + s_{2,2} & 2 + c + s_{2,2} \\ 0 & 2 + c + s_{2,2} & 1 + 2c + s_{2,2} + s_{2,3} \end{pmatrix}.$$

We denote the $(2,2), (2,3), (3,3)$ entries of x'_2 by $s'_{2,2}, c', s'_{2,3}$ respectively.

We choose an element $u \in \mathcal{O}$ such that $\text{tr}(u) = -1, \|u\| = 0$. For example, we can choose $u = \left(\frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, 0\right) \in \mathcal{M}(2) \oplus \mathcal{M}(2) = \mathcal{O}$. Then $u^2 = -u$. By applying $B_2(u)$, x_1 does not change and x'_2 becomes

$$\begin{aligned} x''_2 = B_2(u)(x'_2) &= \begin{pmatrix} 1 + \text{tr}(u) & 2 + \frac{1}{2}s'_{2,2}\bar{u} + \frac{1}{2}c'\bar{u}^2 & \frac{1}{2}c'\bar{u} \\ 2 + \frac{1}{2}s'_{2,2}u + \frac{1}{2}c'u^2 & s'_{2,2} + c'\text{tr}(u) & c' + s'_{2,3}u \\ \frac{1}{2}c'u & c' + s'_{2,3}\bar{u} & s'_{2,3} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2 + \frac{1}{2}(s'_{2,2} - c')\bar{u} & \frac{1}{2}c'\bar{u} \\ 2 + \frac{1}{2}(s'_{2,2} - c')u & s'_{2,2} - c' & c' + s'_{2,3}u \\ \frac{1}{2}c'u & c' + s'_{2,3}\bar{u} & s'_{2,3} \end{pmatrix}. \end{aligned}$$

We denote the $(1,2)$ -entry of x''_2 by $x''_{2,3}$. Then

$$\|x''_{2,3}\| = \left\| 2 + \frac{1}{2}(s'_{2,2} - c')\bar{u} \right\| = 4 + (s'_{2,2} - c')\text{tr}(u) = 4 - s'_{2,2} + c'.$$

Since $s'_{2,2} = 6 + s_{2,2}, c' = 2 + c + s_{2,2}$, we have $\|x_{2,3}\| = c \neq 0$. Therefore, we may assume that x is in the form:

$$x = (x_1, x_2) = \left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_{2,3} & \overline{x_{2,2}} \\ \overline{x_{2,3}} & s_{2,2} & x_{2,1} \\ x_{2,2} & \overline{x_{2,1}} & s_{2,3} \end{pmatrix} \right)$$

where $x_{2,3} \in \mathcal{O}^\times$.

We apply Lemma 6.31 to νx_2 . Note that $\nu\Lambda = \Lambda$. By Lemma 6.31, there exists $g \in G_{1\Lambda k}$ such that the $(2,3)$ and $(3,3)$ entries of $g(\nu x_2)$ are 1 and 0 respectively. So, by applying ν to $g(\nu x_2)$, we may assume that x is in the form:

$$\left(\begin{pmatrix} & & 1 \\ & -2 & \\ 1 & & \end{pmatrix}, \begin{pmatrix} 0 & 1 & \overline{x_{2,2}} \\ 1 & s_{2,2} & x_{2,1} \\ x_{2,2} & \overline{x_{2,1}} & s_{2,3} \end{pmatrix} \right).$$

By applying $B_1(-2\overline{x_{2,2}})$, we may assume that $x_{2,2} = 0$. By applying an element of the form $n_{31}(u)$ where $\text{tr}(u) = 0$, Lemma 6.27 implies that we may assume that $x_{2,1} \in k$. Therefore, x is in the form:

$$x = \left(\begin{pmatrix} & & 1 \\ & -2 & \\ 1 & & \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & s_{2,2} & x_{2,1} \\ 0 & x_{2,1} & s_{2,3} \end{pmatrix} \right)$$

where $s_{2,2}, x_{2,1}, s_{2,3} \in k$. Then $x \in \mathcal{V}_k^{\text{ss}}$ and $\text{Zero}_{\mathbb{P}^2}(x)$ has a rational point $(1, 0, 0)$. Therefore, the element of $H^1(k, \mathfrak{S}_4)$ which corresponds to x now comes from an element of $H^1(k, \mathcal{S})$. This completes the proof of Proposition 6.12. \square

Since Proposition 6.12 is proved, the proof of Theorem 1.19 is finished now. \square

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